ACTA TECHNICA CORVINIENSIS
BULLETIN OF ENGINEERING

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BOUNDARY ADAPTED SPECTRAL APPROXIMATION FOR SPATIAL STABILITY OF BATCHELOR VORTEX

Abstract:
The main goal of this paper is to develop a methodology for analyzing the non-axisymmetrical swirling flows with helical vortex breakdown by means of linear stability analysis. For the case of high Reynolds numbers the eigen value problem governing the linear stability analysis of the Batchelor vortex is investigated using a boundary adapted spectral collocation technique. A symmmetrization is performed eliminating all geometric singularities on the left-hand sides of the governing equations set. The method provides a fairly accurate approximation of the spectrum without any scale resolution restriction.

Keywords:
swirling flow, Batchelor vortex, spectral collocation

INTRODUCTION

Most of the vortex stability analyses concerned axisymmetrical vortices with axial flow [1] in order to explain the vortex breakdown phenomenon observed experimentally for the first time on delta wings [2], in pipes [3] and in cylinders with rotating ends [4]. Obviously, the axial symmetry hypothesis is a major simplification having the main benefit of dramatically reducing the computational cost [5]. On the other hand, it introduces important limitations as far as the three-dimensionality and unsteadiness of the flow are concerned.

The present paper focused on developing an analytical and numerical technique for analyzing the eigenvalue problem governing the linear stability of an inviscid swirling fluid flow under small perturbations. This problem is characterized by a system of ordinary differential equations with variable coefficients.

In most cases, the spatially or temporal stability (classified for open flows as in [6]) under infinitesimal perturbations is reduced to the study of an algebraic eigenvalue problem of this type. The study leads to a dispersion relation connecting in fact the growth rate ω and the axial wavenumber k as a consequence of the condition that nontrivial eigenvalues to exist. Most of the investigations [1], [7] concerned the values of these nondimenional parameters for which the vortex become unstable in the case of either a spatial stability or temporal stability investigation. Since the investigation of this aspect may imply a large amount of measurement, one must resort also to numerical techniques. Although a spatial stability analysis implies the investigation of a nonlinear eigenvalue problem this type of analysis directly provides the frequency ranges of the most unstable modes. In this paper we consider a more general mathematical model for swirling flow stability analysis, starting with the unsteady Euler equations in cylindrical coordinates. In doing so, we can examine both unsteady and circumferentially variable perturbations.

The paper is organized as follows. The eigenvalue problem governing the linear stability analysis for inviscid swirling flows against normal mode perturbations is defined in Section 2. The third section a new radial spectral approximation is proposed and in Section 4 the method is applied for the Batchelor vortex case and the actual numerical procedure is presented. The main advantages of the proposed methods are pointed out in Section 5.

PROBLEM FORMULATION

The governing equations in the case of incompressible and inviscid flow are the Euler equations

$$\nabla \cdot \underline{V} = 0,\ \frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \nabla)\underline{V} = -\frac{1}{\rho}\nabla p \qquad (1)$$

The following flow fields decomposition are used: velocity $\underline{V} + \underline{v}$, pressure $p + \pi$ where $(\underline{V}, p)$ is the base flow, and $(\underline{v}, \pi)$ is the perturbation considered small.

Since the base flow obey the Euler equations (1) the evolution of such small perturbations of the basic flow is governed by the linearized Euler equations

$$\nabla \cdot \underline{v} = 0,\ \frac{\partial \underline{v}}{\partial t} + (\underline{V} \cdot \nabla)\underline{v} + (\underline{v} \cdot \nabla)\underline{V} = -\frac{1}{\rho}\nabla \pi \qquad (2)$$

In the linearization process the second order terms in the small perturbations were neglected. Assuming a steady columnar flow the velocity profile is written

$$\underline{V}(r) = [U(r), 0, W(r)] \qquad (3)$$

where U represents the axial velocity component W the azimuthal component of the velocity both depending only on radius. Next, we consider the following factorization of the small perturbations

$$[\underline{v}(t,z,r,\theta), \pi(t,z,r,\theta)] = [F(r), iG(r), H(r), P(r)] \exp[i(kz + m\theta - \omega t)] \qquad (4)$$

Introducing the factorization form (4) into the linearized Euler equations (2) we obtain the following system of first order differential equations

$$k r F + G + r G' + m H = 0 \qquad (5a)$$

$$kUG - \omega G + \frac{mWG}{r} + \frac{2WH}{r} - P' = 0 \qquad (5b)$$

$$rHkU - rH\omega + m(HW + P) + WG + rGW' = 0 \qquad (5c)$$

$$FkU - F\omega + \frac{FmW}{r} + U'G + kP = 0 \qquad (5d)$$

where prime denotes differentiation with respect to the radius. This homogenous first order differential system is completed with the following boundary conditions at axis and the far field

$$\begin{cases} G(0) = H(0) = 0, F(0), P(0)\ \text{finite}, (m = 0), \\ H(0) \pm G(0) = 0, F(0) = P(0) = 0, (m = \pm 1), \\ F(0) = G(0) = H(0) = P(0) = 0, (|m| > 1), \\ F, G, H, P \to 0, (r \to \infty) \end{cases} \qquad (6)$$

Equations (5) and (6) represent an eigenvalue problem.

BOUNDARY ADAPTED RADIAL SPECTRAL APPROXIMATION

The pseudospectral - collocation method is one of the most used technique for the numerical investigations in hydrodynamic stability problems. Many researchers have demonstrated the applicability of this method with high degree of accuracy to eigenvalue problems governing the linear stability of swirling flows [9-11].

The difference between the classical method and the modified version proposed here is given by the selected spaces involved in the discretization process motivated by the need to adapt the grid points to the singularities of the underlying solution.

In fact the boundary conditions (6) at infinity are applied at a truncated radius distance r_{max} selected large enough such that the numerical results do not depend on this truncated distance.

Following [9] we define the boundary-adapted functions $\{\phi_k\}, k = 1\ldots, N$ of modal type, i. e. each function provides one particular pattern of oscillation

$$\phi_k(r) = \left(1 - \frac{r}{r_{max}}\right) \cdot r \cdot T_k^*(r), \quad \{\phi_k\}, k = 1\ldots, N \qquad (7)$$

with T_k^* the shifted Chebyshev polynomials on $[0, r_{\max}]$. These type of polynomials defined on the physical space are used in order to optimize the interpolative procedure. The choice is based on the condition that the values of the grid points are given by the same elementary analytic expression for all values of N and they did not have to be computed numerically for every N.

The linear transformation that mapps the standard interval $\xi \in [-1,1]$ into the physical range of our problem $r \in [0, r_{\max}]$ and preserves the clustering rate of collocation nodes is defined by the linear transformation

$$r(\xi) = \frac{r_{max}}{2}\xi + \frac{r_{max}}{2} \qquad (8)$$

while the inverse transformation is defined

$$\xi(r) = 2\frac{r}{r_{max}} - 1 \qquad (9)$$

The proposed method allowed us to discard the first and last collocation nodes, expansion functions satisfying the boundary conditions from the construction of our modal boundary-adapted basis. In this way the critical singularities which occurred in evaluating terms like 1/r for the numerical treatment of the eigenvalue problem were eliminated. Then the solution is approximated with respect to this expansion set of functions,

$$(F,G,H,P) = \sum_{k=1}^{N}(u_k, v_k, w_k\, p_k)\phi_k(r) \qquad (10)$$

A modified Chebyshev Gauss grid $\Xi = (\xi_j)_{0 \le j \le N-1}$ in $[-1,1]$ was constructed

$$\xi_j = \cos\left(\pi + \frac{2j\pi}{2N-2}\right),\ \xi_j \in [-1,1],\ j = 0..N-1 \qquad (11)$$

In our case the collocation nodes clustered near the boundaries diminishing the negative effects of the Runge phenomenon. Another aspect is that the convergence of the interpolant on the clustered grid towards unknown function is extremely fast.

Each of the basis functions from (7) meet the relations

$$\phi_k(r_1 = 0) = \phi_k(r_N = r_{\max}) = 0,$$
$$\phi_k(r_j) \ne 0,\ j = 1..N,\ k = 2..N-1 \qquad (12)$$

which implies that each functions F, G, H, P satisfy the boundary conditions (6).

With (10) the mathematical model takes the form

$$kr\sum_{k=1}^{N}u_k\Phi_k(r) + \sum_{k=1}^{N}v_k\Phi_k(r) + rG' + m\sum_{k=1}^{N}w_k\Phi_k(r) = 0 \qquad (13a)$$

$$\left(ku_z - \omega + \frac{mu_\theta}{r}\right)\sum_{k=1}^{N}v_k\Phi_k(r) + \frac{2u_\theta}{r}\sum_{k=1}^{N}w_k\Phi_k(r) - P' = 0 \qquad (13b)$$

$$(rku_z - r\omega)\sum_{k=1}^{N}w_k\Phi_k(r) + m\left(u_\theta\sum_{k=1}^{N}w_k\Phi_k(r) + \sum_{k=1}^{N}p_k\Phi_k(r)\right)$$
$$+ (u_\theta + ru_\theta')\sum_{k=1}^{N}v_k\Phi_k(r) = 0 \qquad (13c)$$

$$\left(ku_z - \omega + \frac{mu_\theta}{r}\right)\sum_{k=1}^{N}u_k\Phi_k(r)$$
$$+ u_z'\sum_{k=1}^{N}v_k\Phi_k(r) + k\sum_{k=1}^{N}p_k\Phi_k(r) = 0 \qquad (13d)$$

Let us denote by $[r] = diag(r_i)$, r_i given by (8), $i = 0,\dots,N-1$, $[\phi] = (\phi_{ij})_{1\le i\le N,\ 1\le j\le N}$, $\phi_{ij} = \phi_j(r_i)$, $[U] = diag(U(r_i))$, $[W] = diag(W(r_i))$, $1 \le i \le N$. The system (13) can be written $(kM_k + \omega M_\omega + mM_m + M_0)\bar{s} = 0$ with $\bar{s} = (\bar{f}\ \ \bar{g}\ \ \bar{h}\ \ \bar{p})^T$ and the matrices M_k, M_ω, M_m and M_0 having the following explicit forms

$$M_k = \begin{pmatrix} [r][\phi] & 0 & 0 & 0 \\ 0 & [U][\phi] & 0 & 0 \\ 0 & 0 & [rU][\phi] & 0 \\ [U][\phi] & 0 & 0 & [\phi] \end{pmatrix} \qquad (14);$$

$$M_\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -[\phi] & 0 & 0 \\ 0 & 0 & -[r][\phi] & 0 \\ -[\phi] & 0 & 0 & 0 \end{pmatrix} \qquad (15)$$

$$M_m = \begin{pmatrix} 0 & 0 & [\phi] & 0 \\ 0 & \left[\frac{W}{r}\right][\phi] & 0 & 0 \\ 0 & 0 & [W][\phi] & [\phi] \\ \left[\frac{W}{r}\right][\phi] & 0 & 0 & 0 \end{pmatrix} \quad (16);$$

$$M_0 = \begin{pmatrix} 0 & [\phi]+[r]D & 0 & 0 \\ 0 & 0 & 2\left[\frac{W}{r}\right][\phi] & -D \\ 0 & [W][\phi]+[rW'][\phi] & 0 & 0 \\ 0 & [U'][\phi] & 0 & 0 \end{pmatrix} \quad (17)$$

By differentiating (10) results

$$F'(r) = \sum_{k=1}^{N} \left\{ \begin{aligned} &\left(1 - \frac{2r}{r_{\max}}\right) u_k T_k^*(r) \\ &+ \left(1 - \frac{r}{r_{\max}}\right) r u_k T_k^{*\prime}(r) \end{aligned} \right\} \quad (18)$$

For $k = 1$ we have $T_1^{*\prime}(r) = 0$ and rewriting (18) we have

$$F'(r) = \left(1 - \frac{2r}{r_{\max}}\right) u_1 T_1^*(r) + \sum_{k=2}^{N} \left\{ \left(1 - \frac{2r}{r_{\max}}\right) u_k T_k^*(r) + \left(1 - \frac{r}{r_{\max}}\right) r u_k T_k^{*\prime}(r) \right\} \quad (19)$$

The shifted Chebyshev polynomials meet the recurrence relation

$$T_n^{*\prime}(r) = \frac{r_{\max}}{4} \frac{(n-1)}{r(r_{\max}-r)} \left[T_{n-1}^*(r) - T_{n+1}^*(r)\right], \; n \ge 2 \quad (20)$$

thus

$$F'(r) = \left(1 - \frac{2r}{r_{\max}}\right) u_1 T_1^*(r) + \sum_{k=2}^{N} u_k \left\{ \left(1 - \frac{2r}{r_{\max}}\right) T_k^*(r) + \left(1 - \frac{r}{r_{\max}}\right) r \frac{r_{\max}}{4} \frac{(k-1)}{r(r_{\max}-r)} \left[T_{k-1}^*(r) - T_{k+1}^*(r)\right] \right\} \quad (21)$$

The interpolant derivative matrix D from (17) was evaluated by

$$D = \begin{pmatrix} \left(1 - \frac{2r_1}{r_{max}}\right) T_1^*(r_1) & E_2(r_1) & E_3(r_1) & \dots & E_N(r_1) \\ \left(1 - \frac{2r_2}{r_{max}}\right) T_1^*(r_2) & E_2(r_2) & E_3(r_2) & \dots & E_N(r_2) \\ \dots & \dots & \dots & \dots & \dots \\ \left(1 - \frac{2r_N}{r_{max}}\right) T_1^*(r_N) & E_2(r_N) & E_3(r_N) & \dots & E_N(r_N) \end{pmatrix} \quad (22)$$

where

$$E_k(r) = \left(1 - \frac{2r}{r_{\max}}\right) T_k^*(r) + \frac{k-1}{4}\left[T_{k-1}^*(r) - T_{k+1}^*(r)\right], \quad k \ge 2 \quad (23)$$

This algorithm allows us to obtain the eigenvalue, the eigenvector, the index of the most unstable mode, the maximum amplitude of the most unstable mode and the critical distance where the perturbation is the most amplified.

The main advantages of the proposed method consist in reducing the computational time by reducing the matrices order to $(4N-8)^2$ and for a certain spectral parameter N we obtain an exponential decreasing error.

NUMERICAL RESULTS FOR BATCHELOR VORTEX

The above presented method was tested on a particular benchmark model: the Batchelor or q-vortex [7].

The flow field is characterized by the velocity field $\underline{V}(r) = [U(r), 0, W(r)]$ [4],

$$U(r) = a + e^{-r^2}, W(r) = \frac{q}{r}(1 - e^{-r^2}) \quad (24)$$

where q represents the swirl number defined as the angular momentum flux divided by the axial momentum flux times the equivalent nozzle radius and a provides a measure of free-stream axial velocity.

In [7] the numerical investigation of the two-point boundary value problem was based on a shooting method. The properties of the Batchelor vortex are pointed out by considering them as functions of the swirl ratio q and the external flow parameter a.

The computed spectrum of the eigenvalue problem is depicted in Figure 1. Graphical representations of the spatial eigenfunction amplitudes of the most unstable mode are given *in the Figure 2. For a stabilization of the Gibbs phenomenon a Lanczos type σ factor [10] was used,*

$$\sigma_k = \frac{\sin\frac{2\pi k}{N}}{\frac{2\pi k}{N}}, \; 1 \le k \le N \qquad (25)$$

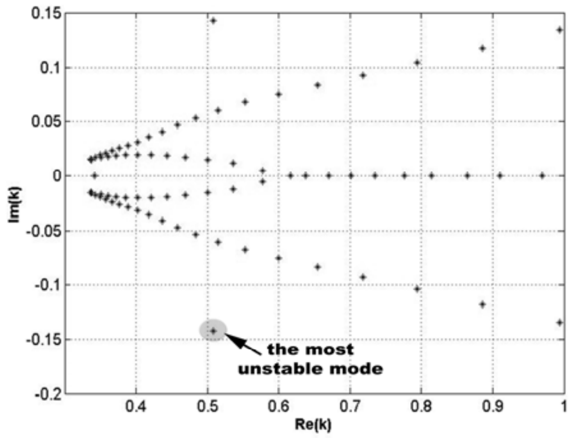

Figure 1. Spectra of the hydrodinamic eigenvalue problem computed at ω = 0.01, m=-3, a = 0, q = 0.1, N = 150.

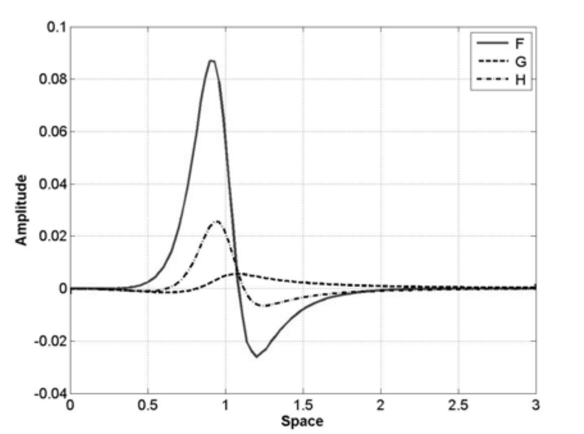

Figure 2. Values of eigenfunction amplitudes of the most unstable mode ω = 0.01, m = -3, a = 0, q = 0.1, N = 150, k = 0.50842-0.14243i.

Table 1. Convergence behaviour of the critical distance for the most unstable mode with ω = 0.01, a = 0, q = 0.1 and m = -3.

N	Axial wavenumber k	Critical distance r_c
100	0.64887-3.7433i	0.00302
150	0.50842-0.14243i	0.90051
180	0.50847-0.14232i	0.92294
250	0.50854-0.14216i	0.95451
300	0.50857-0.14209i	0.93874

CONCLUSION

In this paper we developed a spectral numerical procedure to investigate the spatial stability of a swirling flow subject to infinitesimal perturbations. Using a spectral collocation technique our numerical procedures directly provided relevant information on perturbation amplitude for stable or unstable induced modes, the maximum amplitude of the most unstable mode and the critical distance where the perturbation is the most amplified.

The accuracy of the methods is assessed underlying the necessity for the construction of a certain class of orthogonal expansions functions satisfying the boundary conditions. The key issue was the choice of the grid and the choice of the modal trial basis, the scheme based on shifted Chebyshev polynomials provided good results.

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