

SOME GENERALISATION OF THE BARTLE, DUNFORD AND SCHWARTZ INTEGRABILITY MODEL

■ Abstract:

In [4] the author introduces the notion of pseudosubmeasure as generalization of the submeasure concept [2], and studies some proprieties of the pseudosubmeasure functions with values in a pseudometric space.

The purpose of this paper is to develop an integration theory for these functions, with respect to a semigroup valued measure, using families of pseudosubmeasure and the associated topological rings. AMS Subject Classification Code (2000):28A20, 28B15

■ Keywords:

Bartle, Dunford and Schwartz integrability model, generalisation, theory

■ PRELIMINARIES

The notions and the notations used here follow the paper [4].

Let D be an ordered set with the smallest element d_0 . On this set we define a mapping:

$(d_1, d_2) \rightarrow d_1 + d_2$ with the following properties:

$$(P1) \quad d_0 + d = d + d_0; \forall d \in D$$

$$(P2) \quad d_1 + d_2 = d_2 + d_1; \forall d_1, d_2 \in D$$

$$(P3) \quad d_1 \leq d_2 \Rightarrow d + d_1 \leq d + d_2; \forall d \in D$$

There exists a subset $D_1 \subseteq D$ left directed such that

$$(P4) \quad \forall d \in D_1, \exists d_1 \in D$$

so that $d_1 + d_1 \leq d$.

Definition 1.1. A pseudometric on a set X is a D -valued function $p : X \times X \rightarrow D$ so that:

$$(i) \quad p(x, y) = d_0 \Leftrightarrow x = y$$

$$(ii) \quad p(x, y) = p(y, x), x, y, z \in X$$

$$(iii) \quad p(x, y) \leq p(x, z) + p(z, y); x, y, z \in X.$$

A set X together with a pseudometric ρ is called a pseudometric space and is denoted by (X, ρ, D) .

Remark 1.2. Every uniform space (X, \mathcal{U}) is pseudosemimetrizable, [4]. Let \mathcal{S} be a ring (or algebra) of subsets of fixed set S .

Definition 1.3. A pseudosubmeasure on a ring $\mathcal{S} \subseteq \mathcal{P}(S)$ is a mapping $\gamma : \mathcal{S} \rightarrow D$ such that:

$$(S_1) \quad \gamma(\emptyset) = d_0$$

$$(S_2) \quad E \subseteq F \Rightarrow \gamma(E) \leq \gamma(F), E, F \in \mathcal{S}$$

$$(S_3) \quad \gamma(E \cup F) \leq \gamma(E) + \gamma(F), E, F \in \mathcal{S}$$

If γ has the property that $\gamma(A) = d_0 \Rightarrow A = \emptyset$, then mapping $p : \mathcal{S} \times \mathcal{S} \rightarrow D : p(A, B) = \rho(A \Delta B)$ is a pseudometric on \mathcal{S} invariant to translation Δ (symmetric difference).

Let $\Gamma = \{\gamma_i : \mathcal{S} \rightarrow D\}_{i \in I}$ be a family of pseudosubmeasure on $\mathcal{S} \subseteq \mathcal{P}(S)$ and consider the family $\Omega_\Gamma = \{v_{K,d} : K = \text{finite} \subseteq I, d \in D_1\}$, where $v_{K,d} = (A \in \mathcal{S} : \gamma_i(A) \leq d, i \in K)$.

Then there exist a FN-topology $\tau(\Gamma)$ on \mathcal{S} so that $\mathcal{S}(\Gamma) = (\mathcal{S}, \Delta, \cap, \tau(\Gamma))$ is a topical ring. Let (X, ρ, D) be a pseudometric space.

By generalizing the model established in [3], we introduce an uniform structure on X^S in the following way: To every $K = \text{finite} \subset I, d \in D$, we associate the set:

$$\mathcal{W}_K(D) = \{(f, g) \in X^S \times X^S; \gamma_i \{s \in S; \rho(f(s), g(s)) \geq d\} < K, i \in K\}$$

Then, the family $\{W_k(d); d \in D, K = \text{finite} \subset I\}$ forms a base for an uniform structure \mathcal{U}_Γ on X^S . We denote $X^S(\Gamma) = (X^S, \mathcal{U}_\Gamma)$. The map $f \in X^S$ is a \mathcal{S} -step function if there exists $x_i \in X, E_i \in \mathcal{S}, i = 1, 2, \dots, n$

$$x_i \neq x_j, E_i \cap E_j = \emptyset, i \neq j, \mathcal{S} = \bigcup_{i=1}^n E_i \text{ so that } \forall s \in E_i \text{ imply } f(s) = x_i, i = 1, 2, \dots, n.$$

The space of \mathcal{S} -step functions will be denoted by $\mathcal{E}(\mathcal{S}, X)$.

Definition 1.4. The function $f \in X^S$ is Γ -pseudosubmeasurable if f belongs to the closure of $\mathcal{E}(\mathcal{S}, X)$ in $X^S(\Gamma)$.

We denote by $\mathcal{M}[\mathcal{S}, \Gamma, X]$ the set of these functions.

Definition 1.5. Let $\{f_a\}$ be a generalized sequence in $\mathcal{M}[\mathcal{S}, \Gamma, X]$ and $f \in \mathcal{M}[\mathcal{S}, \Gamma, X]$. If $f_a \rightarrow f$ in $X^S(\Gamma)$, then $\{f_a\}$ converges to f in Γ -pseudomeasures and we denote $f_a \xrightarrow{r} f$.

BASIC ASSUMPTIONS

Let S be a nonempty set, $\mathcal{S} \subset P(S)$ be an algebra of subsets of S and consider a family of pseudosubmeasures $\Gamma = \{\gamma_i : \mathcal{S} \rightarrow D\}_{i \in I}$.

Let $(X_i, \rho_i, D^i), i = 1, 2, 3$ be three pseudometric abelian semigroups for which the addition is uniformly continuous with respect to the pseudometric ρ_i .

In the sequel we consider an additive set function $\mu : \mathcal{S} \rightarrow X_2, \mu(\emptyset) = 0$, and we will choose a family of pseudosubmeasures as it will be specified.

The maps which are to be integrated with respect to μ will belong to X_1^S and the integral will take values in X_3 or its completion \hat{X}_3 .

Suppose that a separate continuous bilinear map exists $X_1 \times X_2 \rightarrow X_3; (x, y) \mapsto x \cdot y$ so that:

$$\begin{aligned} i) \quad & x \cdot 0 = 0 \cdot y = 0, (x \in X_1, y \in X_2) \\ & (x_1 + x_2) \cdot (y_1 + y_2) = x_1 \cdot y_1 + x_1 \cdot y_2 \\ & + x_2 \cdot y_1 + x_2 \cdot y_2, (x_1, y_1 \in X_1, x_2, y_2 \in X_2). \end{aligned}$$

Finally we suppose that Γ_μ, μ and the above bilinear map are chosen so that the following continuity axioms are satisfied:

C1) For every $F \in \mathcal{S}$ and every $d' \in D_1^3$ there exists $d' \in D_1^1$ with the following property: for any $n \in N$, if $\rho_1(x_i, y_i) < d', i = 1, 2, \dots, n$ and $\{E_i\}$ is sequence of pairwise disjoint set from \mathcal{S}

$$\text{then: } \rho_3 \left(\sum_{i=1}^n x_i \mu(E_i \cap F), \sum_{i=1}^n y_i \mu(E_i \cap F) \right) < d.$$

$$C2) \text{ For any } x \in X_1, \lim_{\substack{E \rightarrow \emptyset \\ E \in \mathcal{S}}} x \mu(E) = 0.$$

INTEGRABLE FUNCTIONS

Let $f \in \mathcal{E}(\mathcal{S}, X)$ be a \mathcal{S} -step function.

Definition 3.1. For $E \in \mathcal{S}$, the integral of f on E

$$\text{is by definition } \int_E f d\mu = \sum_{i=1}^n x_i \mu(E_i \cap E).$$

We denote by $\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ the set of Γ_μ -integrable step functions.

Theorem 3.2. (i) Relatively to the operation $(f + g)(s) = f(s) + g(s)$, the space $\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ is a subsemigroup of X_3^3 .

(ii) For $E \in \mathcal{S}$, the map $f \rightarrow \int_E f d\mu$ from

$\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ to X_3 is additive.

(iii) For $f \in \mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ the map $E \rightarrow v(E), v(E) = \int_E f d\mu, E \in \mathcal{S}$ is an additive function.

function.

(iv) For $f \in \mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$:

$$\lim_{E \xrightarrow{\Gamma_\mu} 0} v(E) = \lim_{E \xrightarrow{\Gamma_\mu} 0} \int_E f d\mu = 0$$

The proof follows from definition 3.1 and axioms C_1 and C_2 . The extension of the integral from step functions to the arbitrary functions in X_1^S is based on the following result:

Lemma 3.3. Let $\{f_\alpha\}$ be a generalized sequence from $f \in \mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$, which is Cauchy in

$X_1^S(\Gamma_\mu)$. For $\left\{ \int_E f_\alpha d\mu \right\}$ to be a Cauchy

sequence in X_3 uniform with respect to $E \in \mathcal{S}$ it is necessary and sufficient that:

a) For any neighbourhood \mathcal{V} of 0 in X_3 there exists an index $\alpha_0, K = \text{finite} \subset I$ and $d \in D$,

so that $\alpha \geq \alpha_0$ and $\gamma_i(E) < d, i \in K$ imply

$$\int_E f_\alpha d\mu \in \mathcal{V}$$

b) For any neighbourhood \mathcal{V} of 0 in X_3 there exists an index α_0 and $F \in \mathcal{S}$ so that

$$\int_E f_\alpha d\mu \in \mathcal{V} \text{ if } \alpha \geq \alpha_0 \text{ and } E \in \mathcal{S}, E \subset S - F.$$

Proof. Necessity. For any neighbourhood \mathcal{V} of 0 in X_3 there exists a symmetric entourage W of the uniform structure from X_3 so that $W^2(0) \subseteq \mathcal{V}$.

Let α_0 be so that $\left(\int_E f_\alpha d\mu, \int_E f_{\alpha_0} d\mu \right) \in W$ for any $E \in \mathcal{S}$ if $\alpha \geq \alpha_0$.

From Theorem 3.2., IV, it results that exists $d \in D_1, K = \text{finite} \subset I$ so that we have:

$$\int_E f_{\alpha_0} d\mu \in W(0) \text{ if } \gamma_i(E) < d, i \in K.$$

Therefore $\int_E f_\alpha d\mu \in \mathcal{V}$ if $\alpha \geq \alpha_0$ and $\gamma_i(E) < d, i \in K$, that is the condition a).

The condition b) is obtained by taking $E = \{s \in S : f_{\alpha_0}(s) \neq 0\}$. We have $F \in \mathcal{S}$, and

$$\int_E f_{\alpha_0} d\mu = 0 \text{ for all } E \in \mathcal{S} \text{ with } E \subset S - F.$$

Sufficiency. Let W be a symmetric entourage for X_3 and let $\alpha_0, K = \text{finite} \subset I$, $d \in D_1$ and F be chosen depending on the neighbourhood $W(0)$ according to the conditions a) and b) simultaneously. For F and W , let entourage U from X_1 be chosen according to axiom C_1 .

We write

$$F_{\alpha\alpha'} = \{s \in S; (f_\alpha(s), f_{\alpha'}(s)) \notin U\}, F_{\alpha\alpha'} \in \mathcal{S}.$$

Since $\{f_\alpha\}$ is Cauchy in $X_1^S(\Gamma_\mu)$ there exists $\alpha_1 \geq \alpha_0$ so that $\gamma_i(F_{\alpha\alpha'}) < d, i \in K$ for

$\alpha, \alpha' \geq \alpha_1$. For $E \in \mathcal{S}$ in the semigroup $X_3 \times X_3$, we can write:

$$\begin{aligned} \left(\int_E f_\alpha d\mu, \int_E f_{\alpha'} d\mu \right) &= \left(\int_{E \cap F_{\alpha\alpha'}} f_\alpha d\mu, \int_{E \cap F_{\alpha\alpha'}} f_{\alpha'} d\mu \right) \\ &+ \left(\int_{E \cap (F_{\alpha\alpha'}^c \cap F)} f_\alpha d\mu, \int_{E \cap (F_{\alpha\alpha'}^c \cap F)} f_{\alpha'} d\mu \right) + \\ &\left(\int_{E \cap (F_{\alpha\alpha'}^c \cap F^c)} f_\alpha d\mu, \int_{E \cap (F_{\alpha\alpha'}^c \cap F^c)} f_{\alpha'} d\mu \right) \end{aligned}$$

$$\in W(0) \times W(0) + W(0) \times W(0) + W \subseteq W^2 + W^2 + W^2, \alpha, \alpha' \geq \alpha_1$$

Corollary 3.4. Let $\{f_\alpha\}$ and $\{g_\beta\}$ be two generalized sequences from $\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$, convergent in $X_1^S(\Gamma_\mu)$ to the same function.

If $\left\{ \int_E f_\alpha d\mu \right\}$ and $\left\{ \int_E g_\beta d\mu \right\}$ are generalized

Cauchy sequences in X_3 uniformly in $E \in \mathcal{S}$,

then for any entourage W from X_3 there exists α_0 and β_0 so that if $\alpha \geq \alpha_0$, $\beta \geq \beta_0$ it results

that $\left(\int_E f_\alpha d\mu, \int_E g_\beta d\mu \right) \in W$, uniformly in $E \in \mathcal{S}$.

Proof. Given a symmetric entourage W_1 from X_3 so that $W_1^2 + W_1^2 + W_1^2 \subseteq W$ we choose an entourage U from X_1 corresponding to W_1 according to axiom C_1 .

We write $F_{\alpha\beta} = \{s \in S; (f_\alpha(s), g_\beta(s)) \notin U\}$.

From the previous Lemma it results that there exists $\alpha_0, \beta_0, d \in D, K = \text{finite} \subset I$ so that if

$F \in \mathcal{S}$ and $\alpha > \alpha_0, \beta > \beta_0, \gamma_i(E) < d, i \in K$, $E \subset S - F$, $E \in \mathcal{S}$ we have $\int_E f_\alpha d\mu \in W_1(0)$ and

$$\int_E g_\beta d\mu \in W_1(0)$$

By hypothesis there exist if $\alpha_1 \geq \alpha_0$ and $\beta_1 \geq \beta_0$ so that for $\alpha > \alpha_1, \beta > \beta_1$, we have $\gamma_i(F_{\alpha\beta}) < d, i \in K$.

Expressing the pair $\left(\int_E f_\alpha d\mu, \int_E g_\beta d\mu \right)$ in the same

way as in the proof of the sufficiency from Lemma 3.3., the result is obtained.

Definition 3.5. The function $f \in X_1^S$ is called Γ_μ -integrable if there exists a generalized sequence $\{f_\alpha$ from $\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)\}$ so that $f_\alpha \xrightarrow{\Gamma_\mu} f$ and $\left\{ \int_E f_\alpha d\mu, \right\}$ is a generalized Cauchy sequence in X_3 , uniformly in $E \in \mathcal{S}$. Then the Γ_μ -integral is the element from \hat{X}_3 the completion of X_3 , defined by:

$$\int_E f d\mu = \lim_\alpha \int_E f_\alpha d\mu.$$

From the Corollary 3.4 it results that above Γ_μ -integral is properly defined. We denote by $\mathcal{L}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ the set of Γ_μ -integrable functions from $\mathcal{M}[\mathcal{S}, \Gamma_\mu, X_1]$.

It is obvious that $\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3) \subset \mathcal{L}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ and the Γ_μ -integral restricted to $\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ coincides with the Γ_μ -integral from Definition 3.1.

Theorem 3.6. Relatively to the operation of addition the set $\mathcal{L}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ is a subsemigroup of X_1^S

(i) For $E \in \mathcal{S}$, the mapping $f \rightarrow \int_E f d\mu$ of

$\mathcal{L}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ in \hat{X}_3 is additive:

$$\int_E (f + g) d\mu = \int_E f d\mu + \int_E g d\mu, f, g \in \mathcal{L}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$$

(ii) For $f \in \mathcal{L}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ the mapping

$$E \rightarrow v(E) = \int_E f d\mu, E \in \mathcal{S} \text{ is additive:}$$

$$v\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n v(E_i), E_i \cap E_j = \emptyset, i \neq j, v(\emptyset) = 0$$

For $f \in \mathcal{L}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ we have:

$$\lim_{\Gamma_\mu} v(E) = 0$$

$$E \xrightarrow{E \in \mathcal{S}} \rightarrow$$

The proof follows from Corollary 3.4. and the definition 3.5.

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