BOUNDARY VALUE AND APPLICATION OF CAUCHY’S INTEGRALS ON TWO-DIMENSIONAL ELASTICITY PROBLEM

ABSTRACT: In discussing continuation in two-dimensional elasticity it is necessary to use certain results concerning the boundary values of Cauchy Integrals. The purpose of this paper is to use the value of Cauchy’s Integrals to find the solution problems of two dimensional elasticity.

Keywords: Cauchy’s integral, Boundary values, Limit values, Holder condition

INTRODUCTION

The solution of problems of two-dimensional elasticity by methods using the techniques of the complex variable theory requires the determination from the boundary conditions, of two unknown complex functions, holomorphic at all points in the region of the complex plane occupied by the elastic material \([1]\).

A number of methods of solving these problems have been given by Muskhelishvili (1953) and others. One such method involves the continuation of one or both these unknown functions across the boundary into the region of the complex plane not occupied by elastic material and the expressing the result in terms of Cauchy Integrals. A careful examination of the method and of the results obtained shows that a more simple and direct approach may be given. This is the concern of this paper and simply involves the determination of a holomorphic function having a known real part on the boundary. Such a function may be expressed as a Cauchy’s integral [1].

PRELIMINARY DEFINITIONS

Let \( f(t) \) be an integral function of \( t \) along a line \( L_0 \) then the function

\[
(z) = \frac{1}{2\pi i} \int_{L_0} \frac{f(t)dt}{t-z} \tag{1}
\]

Is called the Cauchy Integral of \( f(t) \) taken along the line \( L_0 \). Clearly \( \varphi(z) \) is an analytic function of \( z \) throughout the whole complex plane except on the line \( L_0 \). Also

\[
(z) = 0(z^{-1}) \text{ at } z = \infty \tag{2}
\]

Further, if the end points A, B of line L are given by \( t = a, b \) respectively, then in general [4]

\[
(z) = \begin{cases} \log(z-a) & \text{at } z = a \\ \log(z-b) & \text{at } z = b \\ \end{cases} \tag{3}
\]

The line L may consist of more than one segment, so there may be several end points. However it is important to know boundary value of Cauchy’s integral when \( z=t_0 \), a point on L. When proceeding along L in the positive neighborhood \( S^+ \) of a point P on it, is that the region to the left of P and the negative neighborhood \( S^- \) is that the right of P. If the region is closed line as \( C \), then \( S \) is the interior of C and \( S^- \) is exterior of C.

Let \( f(t) \) be a function defined at points \( P(t) \) a line \( L \) in the complex plane. The function \( f(t) \) is said to satisfy the Holder condition with constant \( A \) and Holder index \( \mu \), if it satisfies the inequality

\[
|f(t) - f(t_0)| \leq AB |t - t_0|^{\mu} \tag{4}
\]

where \( A > 0, \ 0 < \mu \leq 1 \) in the neighborhood of a point \( t_0 \) of \( L_0 \) [5].

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However it is important to know Boundary Value of Cauchy’s Integral when \( z=t_0 \), a point on \( L_0 \).

Consider the Integral

\[
\frac{1}{2\pi i} \int_{L_0} \frac{f(t)dt}{t-z} \tag{5}
\]

Let \( P_0, P_1 \) be the points \( t_1, t_0 \) of \( L \) in the neighborhood of \( t \) such that

\[
|t_1 - t_0| = |t_0 - t_2| = \delta . \tag{6}
\]

The principal value at \( t_0 \) of Cauchy’s Integral along \( L \) is defined to be:

\[
(t_0) = \frac{1}{2\pi i} \lim_{\substack{\delta \to 0 \\
\delta > 0}} \int_{t_1}^{t_0} \frac{f(t)dt}{t-t_0} . \tag{7}
\]

This may be written

\[
(t_0) = \frac{1}{2\pi i} \lim_{\delta \to 0} \int_{t_1}^{t_0} \frac{f(t)dt}{t-t_0} + \frac{f(t_0)dt}{t-t_0} . \tag{8}
\]

Now, if \( f(t) \) satisfies the Holder Condition [see Appendix], then we will have

\[
\frac{f(t) - f(t_0)}{t-t_0} \leq A |t-t_0|^{\mu} \tag{9}
\]

Since \( 0 < 1 \) then first part of the limit converges. Also

\[
\int_{t_1}^{t_0} \frac{dt}{t-t_0} = \log(t-t_0) | t_0 | + \log(t - t_0) | t_1 | \tag{10}
\]

where any branch of the logarithmic function may be chosen. It is convenient to choose to connect \( P_0, P_1 \) by the arc \( P_0, P_2 \) of a circle in \( S^+ \), with center \( P_0 \) and radius as shown in the figure 1.
If $t = a$, $b$ are the end points $A, B$ respectively of $L$ and the logarithmic function is assumed to change continuously along $A \subset \partial \Omega \subset B$, then
\begin{equation}
\lim_{t \to \pm \infty} \int_a^b \frac{dt}{t} = \log \frac{b-t_a}{a-t_a} + \log \frac{b+t_a}{a+t_a} = \log \frac{b-t}{a-t} + \lim_{t \to \infty} (t - t_a) - \lim_{t \to -\infty} (t - t_a)
\end{equation}
(11)

So
\begin{equation}
(t_a) = \frac{1}{2} f(t_a) + \frac{1}{2m} f(t_a) \log \frac{b-t_a}{a-t_a} + \frac{1}{2m} f(t_a) \left( f(t) - f(t_a) \right) dt.
\end{equation}
(12)

If $L$ is a simple contour $C$, then $a = b$ and

**The Boundary Equation**

Consider elastic material with boundary curve $C$ occupying the region $S$, or the complex plane. For homogeneous isotropic elastic material free from body force, the stress and displacement gradient components may be expressed as the real part of certain combinations of unknown function $\Omega(z)$, $\omega(z)$, and holomorphic at all points $z$ in $S$, and their derivatives. A typical expression of this kind may be written in the form
\begin{equation}
\sum_{r=1}^n P_r(z) \Omega_r(z)
\end{equation}
(14)

where $\Omega_r(z)$, $r = 1, 2, \ldots, n$, denote the unknown holomorphic functions, $\Omega_1(z)$, $\Omega_2(z)$, $\omega_1(z)$, $\omega_2(z)$, $\ldots$, $P_r(z)$, $r = 1, 2, \ldots, n$, denote certain known non-holomorphic function such as $\Omega$, $\omega$ or constants.

If the real part of such a typical expression is given as $f(t)$ on the boundary $C$ where $z = t$ then it is required to determine the $\Omega_r(z)$ such that
\begin{equation}
\text{Re}\left[ \sum_{r=1}^n P_r(z) \Omega_r(z) \right]_{z=t} = f(t)
\end{equation}
(15)

There being given sufficient condition of this form to enable all the $\Omega_r(z)$ to be determined.

This is the problem of two-dimensional elasticity. Now let $\pi_r(z)$, $r = 1, 2, \ldots, n$, be functions, holomorphic at all points $z$ in $S$, having the same boundary values respectively as $P_r(z)$, $r = 1, 2, \ldots, n$ on the boundary $z = t$.

Such that
\begin{equation}
\pi_r(t) = P_r(t)
\end{equation}
(16)
The boundary equation (15) may now be written
\begin{equation}
\text{Re}\left[ \sum_{r=1}^n \pi_r(z) \Omega_r(z) \right]_{z=t} = f(t)
\end{equation}
(17)
Thus the problem has been reduced to that of finding a function, holomorphic at all points $z$ in $S$, which has a known real part on the boundary, $z = t$. This function may be expressed as a Cauchy Integral.

The requirement of equation (16) that the functions $P_r(z)$, $\pi_r(z)$ should have the same value on the boundary greatly restricts the functions $\pi_r(z)$ which is allowed to exist. In practice the functions involved are very simple and present no difficulties.

**The Elastic Half-Plane**

Now consider elastic material with boundary along the real axis occupying the semi-infinite region $S$, $x, y > 0$, of the complex plane. The mean cartesian stress components are given by the equations
\begin{align}
2(\bar{\sigma}x + \bar{\sigma}y) &= \big[ (x^2 - y^2) + 2i(\bar{x}y - z) \big] \Omega(z) + \big[ (x^2 - y^2) - 2i(\bar{x}y - z) \big] \omega(z) \\
2(\bar{\sigma}x - \bar{\sigma}y) &= \big[ (x^2 - y^2) + 2i(\bar{x}y + z) \big] \Omega(z) + \big[ (x^2 - y^2) - 2i(\bar{x}y + z) \big] \omega(z)
\end{align}
(18)

And the mean Cartesian displacement components by
\begin{equation}
\varphi(u + iv) = K \big[ \Omega(z) - \big( x^2 - y^2 \big) \omega(z) \big]
\end{equation}
(19)
where $\Omega(z)$, $\omega(z)$ are the unknown functions, holomorphic at all points $z$ in $S$, which has to be determined from the boundary conditions.

Equation (18), (19) lead to
\begin{align}
4\bar{\sigma}y &= \big[ (x^2 - y^2) + 2i(\bar{x}y - z) \big] \Omega(z) + \big[ (x^2 - y^2) - 2i(\bar{x}y - z) \big] \omega(z) \\
4\bar{\sigma}x &= \big[ (x^2 - y^2) + 2i(\bar{x}y + z) \big] \Omega(z) + \big[ (x^2 - y^2) - 2i(\bar{x}y + z) \big] \omega(z)
\end{align}
(20)

And equation (20) to
\begin{align}
-\frac{\partial \varphi}{\partial x} &= \big[ (x^2 - y^2) + 2i(\bar{x}y - z) \big] \Omega(z) + \big[ (x^2 - y^2) - 2i(\bar{x}y - z) \big] \omega(z) \\
-\frac{\partial \varphi}{\partial y} &= \big[ (x^2 - y^2) + 2i(\bar{x}y + z) \big] \Omega(z) + \big[ (x^2 - y^2) - 2i(\bar{x}y + z) \big] \omega(z)
\end{align}
(21)

Suppose the boundary conditions to be given by any two of the equations
\begin{align}
4(\bar{\sigma}y)_{\nu, \phi} &= f_1(t) \\
4(\bar{\sigma}x)_{\nu, \phi} &= f_2(t)
\end{align}
(22)

Thus it is required to find $\Omega(z)$, $\omega(z)$ to satisfy any two of the equations
\begin{align}
\text{Re}[[x^2 - y^2 + 2i(\bar{x}y - z)] \Omega(z) + (x^2 - y^2 - 2i(\bar{x}y - z)] \omega(z)]_{\nu, \phi} &= f_1(t) \\
\text{Re}[-i[\Omega(z) - \omega(z)]_{\nu, \phi} &= f_2(t)
\end{align}
(23)

These equations contain the holomorphic functions $\Omega_1(z)$, $\Omega_2(z)$, $\omega_1(z)$ and the non-holomorphic function $\Omega$. The essential step in the solution of the problem is to replace $\Omega$ by $z$ which has the same value on the real axis $[4]$.

The boundary equations now become
\begin{align}
\text{Re}[[x^2 - y^2 + 2i(\bar{x}y - z)] \Omega(z) + (x^2 - y^2 - 2i(\bar{x}y - z)] \omega(z)]_{\nu, \phi} &= f_1(t) \\
\text{Re}[-i[\Omega(z) - \omega(z)]_{\nu, \phi} &= f_2(t)
\end{align}
(24)
And the problem reduces to that of finding functions with known real parts on the boundary. The solution is given in terms of Cauchy Integrals by
\begin{equation}
2(\Omega(z) + \Omega(z) + \omega(z)) = \frac{1}{2\pi i} \int_{z_0}^t f(t) \frac{dt}{t-z} + r_z(t)
\end{equation}
(25)
The mean polar stress components are given by [6],

\[ \ddot{r} + \frac{\theta}{a^2} = \Omega(z) + \frac{z}{2} \omega^2(z) \]  

(38)

\[ \ddot{r} - \frac{\theta}{a^2} + 2i/\theta = z \Omega'(z) - \frac{z}{2} \omega'(z) \]  

(39)

These equations lead to

\[ 4r\ddot{r} = Re \left\{ \ddot{\Omega}(z) + \dot{\Omega}'(z) + \frac{z}{2} \omega^2(z) \right\} \]  

(40)

\[ 4r\ddot{\theta} = Re \left\{ \frac{\dot{\Omega}}{z} + \frac{\dot{\Omega}'}{z} + \frac{z}{2} \omega'(z) \right\} \]  

(41)

And equation (20) to

\[ -i\frac{\partial \Phi}{\partial z} + \frac{1}{z} \frac{\partial \Phi}{\partial \theta} = \Omega'(z) + k\Omega(z) - \frac{z}{2} \omega'(z) \]  

(42)

Now let the boundary conditions on the circle \( z = a e^{i\theta} \), be given by any two of the functions

\[ 4\Phi_{\theta\theta} + 4\Phi_{z\theta} = f_{\theta}(t) \]  

(43)

Then it is required to find \( \Omega(z), \omega(z) \) such that

\[ Re \left\{ \ddot{\Omega}(z) - \frac{z}{2} \omega^2(z) \right\} \bigg|_{r=a} = f_{r}(t) \]  

(44)

\[ Re \left\{ -i \ddot{\Omega}(z) + \frac{z}{2} \omega'(z) \right\} \bigg|_{r=a} = f_{\theta}(t) \]  

(45)

\[ Re \left\{ i(1 - k)\Omega(z) - \frac{z}{2} \omega'(z) \right\} \bigg|_{r=a} = f_{r}(t) \]  

(46)

\[ Re \left\{ i(1 + k)\Omega(z) - \frac{z}{2} \omega'(z) \right\} \bigg|_{r=a} = f_{\theta}(t) \]  

(47)

In this case it is sufficient to replace the function \( \frac{z}{2} \) by \( \frac{z^2}{a^2} \) which has the same value on the circle \( z = a e^{i\theta} \) [5].

This then leads to the solution

\[ \ddot{\Omega}(z) - \frac{z}{2} \omega^2(z) = \frac{1}{mi} \int_{t} f_{r}(t) \frac{dt}{t - z} - \frac{1}{2mi} \int_{t} f_{\theta}(t) \frac{dt}{t} + i\lambda \]  

(48)

\[ -i \frac{\dot{\Omega}}{z} - \frac{\dot{\Omega}'}{z} + \frac{z}{2} \omega'(z) = \frac{1}{mi} \int_{t} f_{r}(t) \frac{dt}{t - z} + \frac{1}{2mi} \int_{t} f_{\theta}(t) \frac{dt}{t} + i\lambda \]  

(49)

\[ (1 - k)\ddot{\Omega}(z) - \frac{z}{2} \omega'(z) = \frac{1}{mi} \int_{t} f_{r}(t) \frac{dt}{t - z} - \frac{1}{2mi} \int_{t} f_{\theta}(t) \frac{dt}{t} + i\lambda \]  

(50)

\[ (1 + k)\ddot{\Omega}(z) - \frac{z}{2} \omega'(z) = \frac{1}{mi} \int_{t} f_{r}(t) \frac{dt}{t - z} + \frac{1}{2mi} \int_{t} f_{\theta}(t) \frac{dt}{t} + i\lambda \]  

(51)

where \( \lambda_j \) are constants to be determined to ensure that the stresses are \( o(z^2) \) at infinity.

Any two of these equations will enable \( \Omega(z), \omega(z) \) to be determined.

**CONCLUSIONS**

It has been shown that the solution of the boundary value problem of two-dimensional elasticity may be solved simply and directly by finding holomorphic functions, with known real parts on the boundary, in terms of Cauchy Integrals. The method requires only that simple non-holomorphic functions be replaced by holomorphic functions having the same boundary values and has the advantage that no reference need be made, as in the case of continuation, to those regions of the complex plane not occupied by elastic material. Furthermore this technique provides a straightforward solution to problems in which a combination of any two of the stress or displacement. Components are known on the boundary. In addition the method may be applied to problems which can be solved by conformal transformation.

**REFERENCES**


[3] N. I. Muskhelishvili, Some problems of the mathematical theory of elasticity,

