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BOUNDARY VALUE AND APPLICATION OF CAUCHY'S INTEGRALS ON TWO-DIMENSIONAL ELASTICITY PROBLEM

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ABSTRACT: *In discussing continuation in two-dimensional elasticity it is necessary to use certain results concerning the boundary values of Cauchy integrals. The purpose of this paper is to use the value of Cauchy's Integrals to find the solution problems of two-dimensional elasticity.*
KEYWORDS: *Cauchy's integral, Boundary values, Limit values, Holder condition*

INTRODUCTION

The solution of problems of two-dimensional elasticity by methods using the techniques of the complex variable theory requires the determination from the boundary conditions, of two unknown complex functions, holomorphic at all points in the region of the complex plane occupied by the elastic material [1].

A number of methods of solving these problems have been given by Muskhelishvili (1953) and others. One such method involves the continuation of one or both these unknown functions across the boundary into the region of the complex plane not occupied by elastic material and the expressing the result in terms of Cauchy Integrals. A careful examination of the method and of the results obtained show that a more simple and direct approach may be given. This is the concern of this paper and simply involves the determination of a holomorphic function having a known real part on the boundary. Such a function may be expressed as a Cauchy's Integral [1].

PRELIMINARY DEFINITIONS

Let f (t) be an integral function of t along a line L_o then the function

$$(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} \qquad (1)$$

Is called the Cauchy Integral of f(t) taken along the line L_o Clearly φ(z) is an analytic function of z throughout the whole complex plane except on the line L_o. Also

$$(z) = O(z^{-1}) \quad \text{at} \quad z = \infty \qquad (2)$$

Further, if the end points A, B of line L are given by t = a, b respectively, then in general [4]

$$(z) = \begin{Bmatrix} O\{\log(z-a)\} & \text{at } z = a \\ O\{\log(z-b)\} & \text{at } z = b \end{Bmatrix} \qquad (3)$$

The line L may consist of more than one segment, so there may be several end points. However it is important to know boundary value of Cauchy's integral when z=t_o, a point on L. When proceeding along L in the positive neighborhood S+ of a point P on it, is that the region to the left of P and the negative neighborhood S_- is that the right of P. If the region is closed line as C, then S_+ is the interior of C and S_- is exterior of C.

Let f (t) be a function defined at points P (t) a line L in the complex plane. The function f (t) is said to satisfy the Holder condition, with constant A and Holder index μ, if it satisfies the inequality

$$|f(t) - f(t_0)| \leq AB\,|t - t_0|\,\mu \qquad (4)$$

where A > 0, 0 < μ ≤ 1 in the neighborhood of a point t_o of L_o [5].

BOUNDARY VALUE OF CAUCHY'S INTEGRALS ON ELASTICITY PROBLEM

However it is important to know Boundary Value of Cauchy's Integral when z=t_o, a point on L_o

Consider the Integral

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} \qquad (5)$$

Let P_1, P_2 be the points t_1, t_2 of L in the neighborhood of t such that

$$|t_1 - t_0| = |t_2 - t_0| = \delta\,. \qquad (6)$$

The principal value at t_o of Cauchy's Integral along L is defined to be:

$$(t_0) = \frac{1}{2\pi i}\lim_{\delta \to 0}\left\{\int_{AP_1} + \int_{P_2B}\right\}\frac{f(t)dt}{t-t_0}. \qquad (7)$$

This may be written

$$(t_0) = \frac{1}{2\pi i}\lim_{\delta \to 0}\left\{\int_{AP_1} + \int_{P_2B}\right\}\left\{\frac{f(t)-f(t_0)}{t-t_0} + \frac{f(t_0)}{t-t_0}\right\}dt\,. \qquad (8)$$

Now, If f (t) satisfies the Holder Condition [see Appendix], then we will have

$$\frac{f(t)-f(t_0)}{t-t_0} \leq \frac{A}{|t-t_0|^{1-\mu}} \qquad (9)$$

Since 0 ≤1-μ <, 1 then first part of the limit converges. Also

$$\left\{\int_{AP1} + \int_{P2B}\right\}\frac{dt}{t-t_0} = \left[\log(t-t_0)\right]_a^{t_1} + \left[\log(t-t_0)\right]_{t_2}^{b} \qquad (10)$$

where any branch of the logarithmic function may be chosen. It is convenient to choose to connect P_1, P_2 by the arc P_1P_3 P_2 of a circle in S_+, with center P_0 and radius as shown in the figure 1.

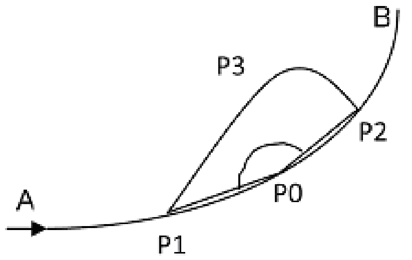

Figure 1.

If $t = a, b$ are the end points A,B respectively of L and the logarithmic function is assumed to change continuously along $A_{P_1P_3P_2}B$, then

$$\lim_{\delta \to o} \left\{ \int_{AP_1} + \int_{P_2B} \right\} \frac{dt}{t \quad t_o} = \log \frac{b - t_o}{a - t_o} + \lim_{\delta \to o} \log \frac{t_1 - t_o}{t_2 - t_o} = \tag{11}$$

$$\log \frac{b - t_o}{a - t_o} + \lim_{\delta \to o} i\theta = \log \frac{b - t_o}{a - t_o} + \lim_{\delta \to o} i\pi.$$

So

$$(t_0) = \frac{1}{2} f(t_0) + \frac{1}{2\pi i} f(t_0) \log \frac{b - t_0}{a - t_0} + \frac{1}{2\pi i} \int \frac{f(t) - f(t_0)}{t - t_0} dt. \tag{12}$$

If L is a simple contour C, then a=b and

THE BOUNDARY EQUATION

Consider elastic material with boundary curve C occupying the region S_+ or the complex plane. For homogeneous isotropic elastic material free from body force, the stress and displacement gradient components may be expressed as the real part of certain combinations of unknown function Ω (z), ω (z), and holomorphic at all points z in S_+ and their derivatives. A typical expression of this kind may be written in the form

$$\sum_{r=1}^{n} P_r(z)\Omega_r(z) \tag{14}$$

where Ω_t (z), r=1,2,...,n , denote the unknown holomorphic functions, Ω'_t (z), Ω''_t (z),..., ω'(z), ω"(z) ,... and P_r (z) , r=1,2,...,n , denote certain known non-holomorphic function such as $\bar{z}$, $\frac{z}{\bar{z}}$ ... or constants.

If the real part of such a typical expression is given as f (t) on the boundary C where z=t then it is required to determine the Ω_t (z) such that

$$\mathrm{Re}\left[\sum_{r=1}^{n} P_r(z)\Omega_r(z)\right]_{z=t} = f(t) \tag{15}$$

There being given sufficient condition of this form to enable all the Ω_r (z) to be determined.

This is the problem of two-dimensional elasticity.

Now let π_t (z), r=1,2,...,n , be functions, holomorphic at all points z in S_+ having the same boundary values respectively as $P_r(z)$, r=1,2,...,n on the boundary z=t .

Such that

$$\pi_{r_+}(t) = P_{r_+}(t) \tag{16}$$

The boundary equation (15) may now be written

$$\mathrm{Re}\left[\sum_{r=1}^{n} \pi_r(z)\Omega_r(z)\right]_{z=t} = f(t) \tag{17}$$

Thus the problem has been reduced to that of finding a function, holomorphic at all points z in S_+ which has a known real part on the boundary, z=t . This function may be expressed as a Cauchy Integral.

The requirement of equation (16) that the functions P_r (z), π_r (z) should have the same value on the boundary greatly restricts the functions π_+ (z) which is assumed to exist. In practice the functions involved are very simple and present no difficulties.

THE ELASTIC HALF-PLANE

Now consider elastic material with boundary along the real axis occupying the semi-infinite region S_+, y>0, of the complex plane. The mean cartesian stress components are given by the equations

$$2(\breve{x}\breve{x} + \breve{y}\breve{y}) = \Omega'(z) + \overline{\Omega'(z)} \tag{18}$$

$$2(\breve{x}\breve{x} - \breve{y}\breve{y} + 2i\breve{x}\breve{y}) = -z\Omega''(z) - \overline{\omega''(z)} \tag{19}$$

And the mean Cartesian displacement components by

$$\delta\mu(u + iv) = k\Omega(z) - z\Omega''(z) - \overline{\omega''(z)} \tag{20}$$

where Ω (z), ω (z) are the unknown functions, holomorphic at all points z in S_+ which has to be determined from the boundary conditions.

Equation (18), (19) lead to

$$4\breve{y}\breve{y} = \mathrm{Re}\{2\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\} \tag{21}$$

$$4\breve{x}\breve{y} = \mathrm{Re} - i\{\bar{z}\Omega''(z) + \omega''(z)\} \tag{22}$$

And equation (20) to

$$-\delta\mu \frac{\partial u}{\partial x} = \mathrm{Re}\{(1 - k)\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\} \tag{23}$$

$$-\delta\mu \frac{\partial v}{\partial x} = \mathrm{Re} - i\{(1 + k)\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\} \tag{24}$$

Suppose the boundary conditions to be given by any two of the equations

$$\left\{ \begin{matrix} 4(\breve{y}\breve{y})_{y=0} = f_1(t) & 4(\breve{x}\breve{y})_{y=0} = f_2(t) \\ -\delta\mu(\frac{\partial u}{\partial x})_{y=0} = f_3(t) & -\delta\mu(\frac{\partial u}{\partial x})_{y=0} = f_4(t) \end{matrix} \right\} \tag{25}$$

Thus it is required to find Ω (z), ω (z) to satisfy any two of the equations

$$\mathrm{Re}[\{2\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_1(t) \tag{26}$$

$$\mathrm{Re}[-i\{\bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_2(t) \tag{27}$$

$$\mathrm{Re}[\{(1 - k)\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_3(t) \tag{28}$$

$$\mathrm{Re}[i\{(1 + k)\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_4(t) \tag{29}$$

These equations contain the holomorphic functions Ω'_t (z), Ω''_t (z), ω" (z) and the non-holomorphic function $\bar{z}$. The essential step in the solution of the problem is to replace $\bar{z}$ by z which has the same value on the real axis [4].

The boundary equations now become

$$\mathrm{Re}[\{2\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_1(t) \tag{30}$$

$$\mathrm{Re}[-i\{\bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_2(t) \tag{31}$$

$$\mathrm{Re}[\{(1 - k)\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_3(t) \tag{32}$$

$$\mathrm{Re}[i\{(1 + k)\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z)\}]_{y=0} = f_4(t) \tag{33}$$

And the problem reduces to that of finding functions with known real parts on the boundary. The solution is given in terms of Cauchy Integrals by

$$2\Omega'(z) + \bar{z}\Omega''(z) + \omega''(z) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} f_1(t) \frac{dt}{t - z} + \tau_1(t) \tag{34}$$

$$-i\{\bar{z}\Omega''(z)+\omega''(z)\} = \frac{1}{\pi i}\int_{-\infty}^{+\infty} f_2(t)\frac{dt}{t-z}+\tau_2(t) \qquad (35)$$

$$(1-k)\Omega'(z)+\bar{z}\Omega''(z)+\omega''(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} f_3(t)\frac{dt}{t-z}+\tau_3(t) \qquad (36)$$

$$i\{(1+k)\Omega'(z)+\bar{z}\Omega''(z)+\omega''(z)\} = \frac{1}{\pi i}\int_{-\infty}^{+\infty} f_4(t)\frac{dt}{t-z}+\tau_{41}(t) \qquad (37)$$

where $\tau_j(z)$, j=1,2,3,4 are arbitrary functions of z which vanish for stresses that are $o(z^{-1})$ at infinity [6].

Hence if any two of the $f_j(t)$, j=1,2,3,4 are known, the functions $\Omega(z)$, $\omega(z)$ may be found and the elastic problem solved.

THE ELASTIC PLATE CONTAINING A CIRCULAR HOLE

Consider elastic material occupying the region S_+ lying outside the circle C, $z\bar{z}=a^2$.

The mean polar stress components are given by [6],

$$\tilde{rr}+\tilde{\theta\theta} = \Omega'(z)+\overline{\Omega'(z)} \qquad (38)$$

$$\tilde{rr}-\tilde{\theta\theta}+2i\tilde{r\theta} = -z\Omega''(z)-\overline{\omega''(z)} \qquad (39)$$

These equations lead to

$$4\tilde{rr} = \mathrm{Re}\left\{2\Omega'(z)+\bar{z}\Omega''(z)+\frac{z}{\bar{z}}\omega''(z)\right\} \qquad (40)$$

$$4\tilde{r\theta} = \mathrm{Re}\,-i\left\{\bar{z}\Omega''(z)+\frac{z}{\bar{z}}\omega''(z)\right\} \qquad (41)$$

And equation (20) to

$$-\frac{\delta\mu i}{z}\frac{\partial}{\partial\theta}(u-iv) = \Omega'(z)-k\overline{\Omega'(z)}-\bar{z}\Omega''(z)-\frac{z}{\bar{z}}\omega''(z) \qquad (42)$$

Now let the boundary conditions on the circle C, $z=t=ae^{i\vartheta}$, be given by any two of the functions

$$\begin{cases} 4(\tilde{yy})_{y=0} = f_1(t) & 4(\tilde{xy})_{y=0} = f_2(t) \\ \left[-\frac{\delta\mu i}{z}\frac{\partial}{\partial\theta}(u-iv)\right]_{r=a} = f_3(t)-if_4(t) & \end{cases} \qquad (43)$$

Then it is required to find $\Omega(z)$, $\omega(z)$ such that

$$\mathrm{Re}\left[\left\{2\Omega'(z)-\bar{z}\Omega''(z)-\frac{z}{\bar{z}}\omega''(z)\right\}\right]_{r=a} = f_1(t) \qquad (44)$$

$$\mathrm{Re}\left[-i\left\{\bar{z}\Omega''(z)+\frac{z}{\bar{z}}\omega''(z)\right\}\right]_{r=a} = f_2(t) \qquad (45)$$

$$\mathrm{Re}\left[\left\{(1-k)\Omega'(z)-\bar{z}\Omega''(z)-\frac{z}{\bar{z}}\omega''(z)\right\}\right]_{r=a} = f_3(t) \qquad (46)$$

$$\mathrm{Re}\left[i\left\{(1+k)\Omega'(z)-\bar{z}\Omega''(z)-\frac{z}{\bar{z}}\omega''(z)\right\}\right]_{r=a} = f_4(t) \qquad (47)$$

In this case it is sufficient to replace the function $\frac{z}{\bar{z}}$ by $\frac{z^2}{a^2}$ which has the same value on the circle $z=t=ae^{i\vartheta}$ [5].

This then leads to the solution

$$2\Omega'(z)-\bar{z}\Omega''(z)-\frac{z^2}{a^2}\omega''(z) = \frac{1}{\pi i}\int_c f_1(t)\frac{dt}{t-z}-\frac{1}{2\pi i}\int_c f_1(t)\frac{dt}{t}+i\lambda_1 \qquad (48)$$

$$-i\left\{\bar{z}\Omega''(z)+\frac{z^2}{a^2}\omega''(z)\right\} = \frac{1}{\pi i}\int_c f_2(t)\frac{dt}{t-z}-\frac{1}{2\pi i}\int_c f_2(t)\frac{dt}{t}+i\lambda_2 \qquad (49)$$

$$(1-k)\Omega'(z)-\bar{z}\Omega''(z)-\frac{z^2}{a^2}\omega''(z) = \frac{1}{\pi i}\int_c f_3(t)\frac{dt}{t-z}-\frac{1}{2\pi i}\int_c f_3(t)\frac{dt}{t}+i\lambda_3 \qquad (50)$$

$$i\left\{(1+k)\Omega'(z)-\bar{z}\Omega''(z)-\frac{z^2}{a^2}\omega''(z)\right\} = \frac{1}{\pi i}\int_c f_4(t)\frac{dt}{t-z}-\frac{1}{2\pi i}\int_c f_4(t)\frac{dt}{t}+i\lambda_4 \qquad (51)$$

where λ_j, j=1,2,3,4 are constants to be determined to ensure that the stresses are $o(z^{-1})$ at infinity.

Any two of these equations will enable $\Omega(z)$, $\omega(z)$ to be determined.

CONCLUSIONS

It has been shown that the solution of the boundary value problem of two-dimensional elasticity may be solved simply and directly by finding holomorphics functions, with known real parts on the boundary, in terms of Cauchy Integrals. The method requires only that simple non-holomorphic functions be replaced by holomorphic functions having the same boundary values and has the advantage that no reference need be made, as in the case of continuation, to those regions of the complex plane not occupied by elastic material. Furthermore this technique provides a straightforward solution to problems in which a combination of any two of the stress or displacement. Components are known on the boundary. In addition the method may be applied to problems which can be solved by conformal transformation.

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