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USING THE GREEN'S FUNCTION METHOD TO ANALYSE THE RESPONSE OF AN INFINITE WIRE ON VISCO-ELASTIC SUPPORT UNDER MOVING LOAD

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ABSTRACT: The paper herein deals with the response of an infinite wire on viscoelastic support under moving harmonic load in order to point out the basic features of the overhead contact wire system (catenary). To this end, the Green's function method is applied. The wire response under a stationary harmonic load is similar to the one of the system with a single degree of freedom, excepting the phase resonance. When the harmonic load moves at sub-critical velocities, the resonance frequency decreases and the wire response becomes higher. At the over-critical velocities, the elastic waves do not propagate in front of the harmonic load.
KEYWORDS: catenary, wire, moving load, Green's function, receptance

INTRODUCTION

For the high speed trains, the pantograph-catenary system is critical for the stable current collection. Indeed, when the train velocity increases, the variation in contact force at the pantograph-catenary interface also increases and many undesirable effects occur: loss of contact, arcing and wear [1, 2].

The pantograph-catenary interaction has been intensively studied in the last 40 years once the trains speed has much increased [3-5]. From mechanical view point, the interaction between a moving pantograph and catenary is a part of the field of the classical 'moving load problem' [6]. In fact, such problems deal with the vibration of a moving sub-system on an elastic structure. Prior to solve this question, it is interesting to highlight the structure's response to a moving load [7, 8].

Many theoretical models of the catenary take into consideration an infinite uniform wire on viscoelastic support (Winkler support). This model is simpler because the influence of the elasticity variation due to the supports and hangers is neglected. The results derived from this model are basic and they can be used for a comparison, as did A. Metrikine [7] in order to highlight the non-linear effect brought about by the hangers.

In this paper, the response of an infinite wire on viscoelastic support due to a harmonic moving load is analyzed using the Green's functions method. This method has been successfully applied by the author to investigate the response of both ballasted and slab track under moving loads [9, 10].

THE MECHANICAL MODEL

It considers the case a simple catenary system (Figure 1), consisting of the equidistant supporters (1), the cable arms (2), the messenger wire (3), the hangers (4) and the wire (5). Both messenger and contact wires are stressed.

Further, the simplest model of catenary will be considered, namely an infinite wire on a continuous elastic support (fig 2). The catenary wire has the linear mass m and it is tensed by the force T. The elastic support contains elastic and of damping elements with linear characteristics, uniformly distributed along the catenary. A harmonic force of amplitude P and angular frequency ▫ acts upon the wire and moves at a constant speed V.

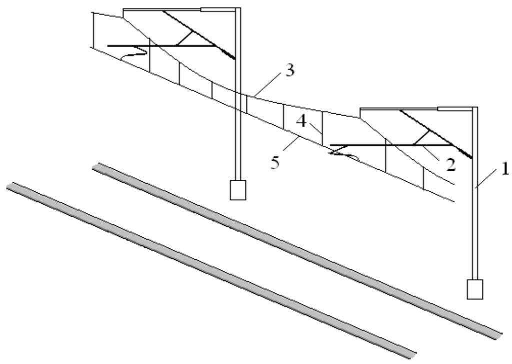

Figure 1. Simple catenary system: 1. supporter; 2. steady arm; 3. messenger wire; 4. hanger; 5. contact wire.

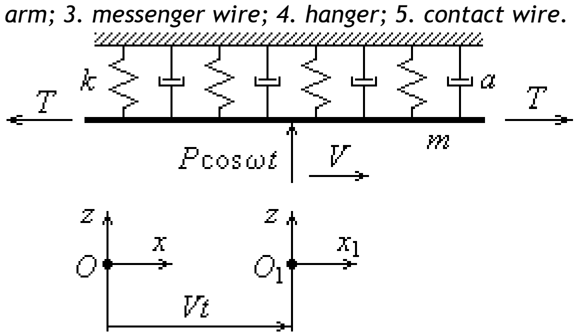

Figure 2. Mechanical model of the catenary system

The wire movement is reported to the fixed system Oxz. Also, a moving system attaches against the force O_1x_1z with

$$x = Vt + x_1 \tag{1}$$

The wire motion equation reported to the fixed referential is

$$m\frac{\partial^2 w}{\partial t^2} + a\frac{\partial w}{\partial t} + kw - T\frac{\partial^2 w}{\partial x^2} = P\cos\omega t\delta(x - Vt), \tag{2}$$

where k is the elastic constant, and a - the damping constant of the continuous elastic support.
Also, the boundary conditions have to be considered

$$\lim_{|x-Vt|\to\infty} w = 0. \tag{3}$$

Actually, the steady state behaviour is interesting and due to that, the change of variable (1) is recommended, where the motion is reported to the moving referential. Practically, the following relations will be applied

$$\frac{\partial^n}{\partial x^n} \to \frac{\partial^n}{\partial x_1{}^n}, \quad \frac{\partial^n}{\partial t^n} \to \left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial x_1}\right)^n. \tag{4}$$

The equation of motion (2) and the boundary conditions (3) become

$$(mV^2 - T)\frac{\partial^2 w}{\partial x_1{}^2} - aV\frac{\partial w}{\partial x_1} - 2mV\frac{\partial^2 w}{\partial x_1 \partial t} +$$
$$+ kw + a\frac{\partial w}{\partial t} + m\frac{\partial^2 w}{\partial t^2} = P\cos\omega t\delta(x_1) \tag{5}$$

$$\lim_{|x_1|\to\infty} w = 0. \tag{6}$$

Considering the steady state harmonic behaviour, the complex variables

$$\overline{w}(x_1,t) = \overline{w}(x_1)e^{\mathrm{i}\omega t}, \quad \overline{P}(t) = \overline{P}e^{\mathrm{i}\omega t} = Pe^{\mathrm{i}0}e^{\mathrm{i}\omega t} \tag{7}$$

have to verify the following equation

$$(mV^2 - T)\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}x_1{}^2} - V(a + 2\omega m\mathrm{i})\frac{\mathrm{d}\overline{w}}{\mathrm{d}x_1} +$$
$$+ (k - \omega^2 m + \omega a\mathrm{i})\overline{w} = \overline{P}\delta(x_1) \tag{8}$$

and the boundary conditions

$$\lim_{|x_1|\to\infty} \overline{w} = 0. \tag{9}$$

To solve the problem defined by the equation (8) and the boundary conditions (9), the Green's functions method may be applied [11]. In fact, the solution is given as

$$\overline{w}(x_1) = \int_{-\infty}^{\infty} G(x_1,\xi)\overline{P}\delta(\xi)\mathrm{d}\xi = \overline{P}G(x_1,0), \tag{10}$$

where $G(x_1, \xi)$ is the Green's function. This function represents the wire response in the x_1 section of the moving reference frame, due to a unit harmonic force applied in the ▫ section of the same moving reference frame. It has to be observed the fact that the wire response is defined by the Green's function and this function is the receptance.
The Green's function can be built as a linear combination of the eigenfunctions of the differential operator of the equation (8). To find this function, the starting point is the homogenous equation

$$(mV^2 - T)\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}x_1{}^2} - V(a + 2\omega m\mathrm{i})\frac{\mathrm{d}\overline{w}}{\mathrm{d}x_1} +$$
$$+ (k - \omega^2 m + \omega a\mathrm{i})\overline{w} = 0 \tag{11}$$

and its solution

$$\overline{w}(x_1) = Ae^{\lambda x_1}. \tag{12}$$

Then, the characteristic equation is obtained

$$(mV^2 - T)\lambda^2 - V(a + 2\omega m\mathrm{i})\lambda + k - \omega^2 m + \omega a\mathrm{i} = 0 \tag{13}$$

After some calculations, this equation takes the following form

$$(V^2 - c^2)\lambda^2 - 2V(\zeta\omega_0 + \omega\mathrm{i})\lambda + \omega_0^2 - \omega^2 + 2\zeta\omega_0\omega\mathrm{i} = 0 \tag{14}$$

where

$$\zeta = \frac{a}{2\sqrt{mk}}, \quad \omega_0^2 = \frac{k}{m}, \quad c = \sqrt{\frac{T}{m}} \tag{15}$$

The solutions of the characteristic equation represent the eigenvalues

$$\lambda_{1,2} = \frac{\omega_0}{c}\cdot\mu_{1,2} \tag{16}$$

where $$\mu_{1,2} = \frac{\alpha(\zeta + \Omega\mathrm{i}) \pm \sqrt{1 - \Omega^2 - \alpha^2(1-\zeta^2) + 2\zeta\Omega\mathrm{i}}}{\alpha^2 - 1} \tag{17}$$

with $$\alpha = \frac{V}{c}, \quad \Omega = \frac{\omega}{\omega_0}. \tag{18}$$

There are two cases, the so-called sub-critical and overcritical cases.

1. The sub-critical case *($\alpha < 1$) - the force velocity is smaller than the velocity of the elastic wave in the contact wire; the critical velocity has value of c. In this case, the eigenvalues real parts have opposite signs*

$$\mathrm{Re}\,\lambda_1 < 0, \quad \mathrm{Re}\,\lambda_2 > 0. \tag{19}$$

In fact, the Green's function $G(x_1, \xi)$ has two forms satisfying the boundary conditions

$$G^-(x_1,\xi) = A^- e^{\lambda_2 x_1} \text{ for } -\infty < x_1 < \xi \tag{20}$$
$$G^+(x_1,\xi) = A^+ e^{\lambda_1 x_1} \text{ for } \xi < x_1 < \infty,$$

where A^- and A^+ depend on the ξ variable. These functions will be calculated using both continuity and jump conditions.
The Green's function has to be continuous in $x_1 = \xi$

$$A^- e^{\lambda_2 \xi} = A^+ e^{\lambda_1 \xi}. \tag{21}$$

Its derivation in respect to x_1 has a jump in $x_1 = \xi$

$$\frac{\partial G^+(\xi+0,\xi)}{\partial x_1} - \frac{\partial G^-(\xi-0,\xi)}{\partial x_1} = \frac{1}{T}\frac{1}{\alpha^2 - 1} \tag{22}$$

respectively

$$\lambda_1 A^+ e^{\lambda_1 \xi} - \lambda_2 A^- e^{\lambda_2 \xi} = \frac{1}{T}\frac{1}{\alpha^2 - 1}. \tag{23}$$

Upon solving the equations (21) and (23), it is obtained

$$A^- = \frac{1}{\sqrt{kT}}\frac{e^{-\sqrt{\frac{k}{T}}\mu_2\xi}}{(\mu_1 - \mu_2)(\alpha^2 - 1)}, \quad A^+ = \frac{1}{\sqrt{kT}}\frac{e^{-\sqrt{\frac{k}{T}}\mu_1\xi}}{(\mu_1 - \mu_2)(\alpha^2 - 1)} \tag{24}$$

and then, the Green's function

$$G^-(x_1,\xi) = \frac{1}{\sqrt{kT}}\frac{e^{\sqrt{\frac{k}{T}}\mu_2(x_1 - \xi)}}{(\mu_1 - \mu_2)(\alpha^2 - 1)} \text{ for } -\infty < x_1 < \xi \tag{25}$$

$$G^{+}(x_1,\xi)=\frac{1}{\sqrt{kT}}\frac{e^{\sqrt{\frac{k}{T}}\mu_1(x_1-\xi)}}{(\mu_1-\mu_2)(\alpha^2-1)}\ \text{for}\ \xi<x_1<\infty .$$

*2. **The overcritical case** (α>1) represents the situation when the harmonic force travels at a higher speed than the wave propagation speed through the contact wire. In this case, the real part of both eigenvalues has positive sign*

$$\mathrm{Re}\lambda_{1,2}>0. \tag{26}$$

The Green's function takes the following forms

$$G^{-}(x_1,\xi)=A_1e^{\lambda_1 x_1}+A_2e^{\lambda_2 x_1}\ \text{for}\ -\infty<x_1<\xi \tag{27}$$

$$G^{+}(x_1,\xi)=0\ \text{for}\ \xi<x_1<\infty ,$$

where A_1 and A_2 depend also on ξ.

The continuity condition of the function and the one of the derivate jump lead to the following equations

$$A_1e^{\lambda_1 \xi}+A_2e^{\lambda_2 \xi}=0 \tag{28}$$

$$\lambda_1\ A_1e^{\lambda_1 \xi}+\lambda_2\ A_2e^{\lambda_2 \xi}=-\frac{1}{T}\frac{1}{\alpha^2-1}.$$

Solving the equations (28), it obtains

$$A_{1,2}=\mp\frac{1}{\sqrt{kT}}\frac{e^{-\sqrt{\frac{k}{T}}\mu_{1,2}\xi}}{(\mu_1-\mu_2)(\alpha^2-1)}. \tag{29}$$

Finally, the Green's function can be written as

$$G^{-}(x_1,\xi)=\frac{1}{\sqrt{kT}}\frac{e^{\sqrt{\frac{k}{T}}\mu_2(x_1-\xi)}-e^{\sqrt{\frac{k}{T}}\mu_1(x_1-\xi)}}{(\mu_1-\mu_2)(\alpha^2-1)} \tag{30}$$

for $-\infty<x_1<\xi$

$G^{+}(x_1,\xi)=0$ *for* $\xi<x_1<\infty$.

The latter can be also written as

$$G(x_1,\xi)=\frac{1}{\sqrt{kT}}\frac{e^{\sqrt{\frac{k}{T}}\mu_2(x_1-\xi)}-e^{\sqrt{\frac{k}{T}}\mu_1(x_1-\xi)}}{(\mu_1-\mu_2)(\alpha^2-1)}H(\xi-x_1) \tag{31}$$

where H(.) is Heaviside's unit step function. Equation (31) shows the fact that in front of the moving harmonic load the wire is not perturbed by the elastic waves.

NUMERICAL APPLICATION

Further on, using the above method based on the Green's function, the results of the numerical simulation derived from a particular wire on viscoelastic support are presented.

The following data have been considered [7]: m=1.1 kg/m, T=15 kN, k=0.4 kN/m^2 and a=0.5 Ns/m^2. In fact, the natural frequency of the wire on viscoelastic support is 3 Hz, the damping degree - 0,012 and the wave propagation speed - 117 m/s.

Figure 3 shows the wire receptance at the point of the stationary unit harmonic force (α = 0) versus the relative angular frequency ($\Omega=\omega/\omega_0$). Three values of the damping degree are considered. As it can be observed, the wire response has a peak similar to the one of a system with single degree of freedom. However, the phase resonance occurs for the π/4 value instead of π/2. Increasing the damping, the receptance becomes lower around the resonance frequency.

Figure 4 shows the influence of the speed of the moving harmonic load on the receptance of the wire at the point of the unit harmonic force. Only the reference value of the damping degree is taken into account (ζ = 0,012). Actually, only the range of the sub-critical speeds is considered in this simulation. It can be seen that by increasing the speed of moving harmonic load, the resonance frequency of the wire decreases. In addition, the receptance of the wire increases around resonance.

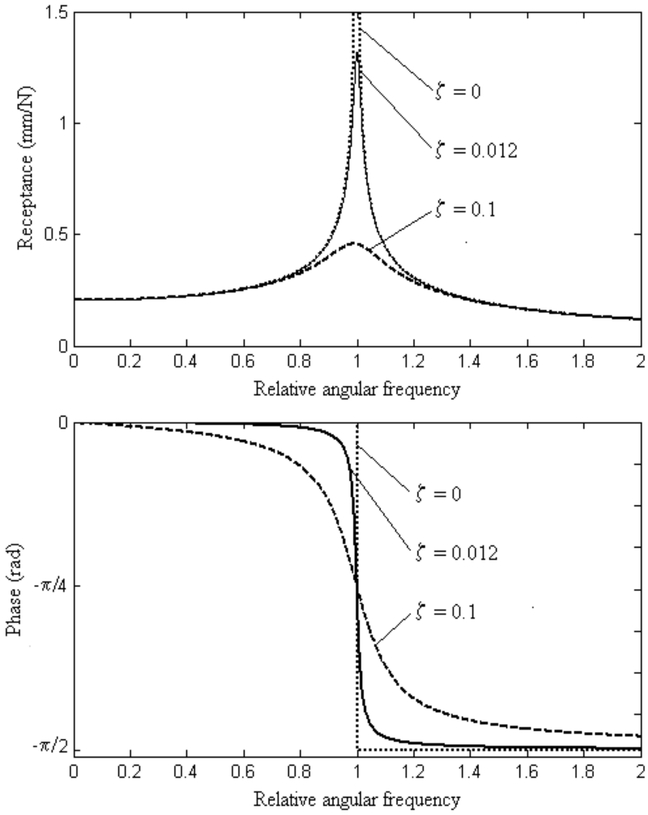

Figure 3. Wire response due to a stationary harmonic force: (a) receptance modulus; (b) recepance phase.

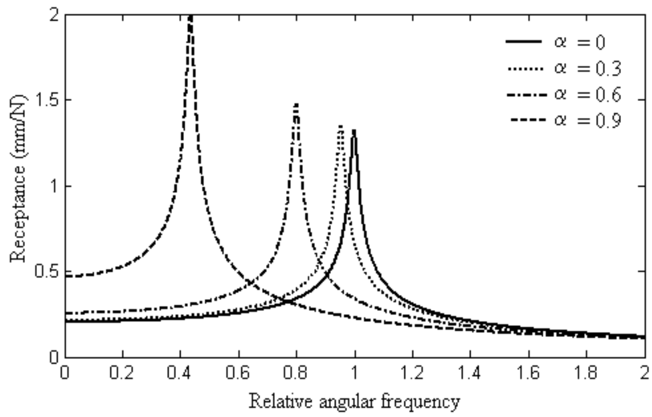

Figure 4. Influence of the speed on the wire response

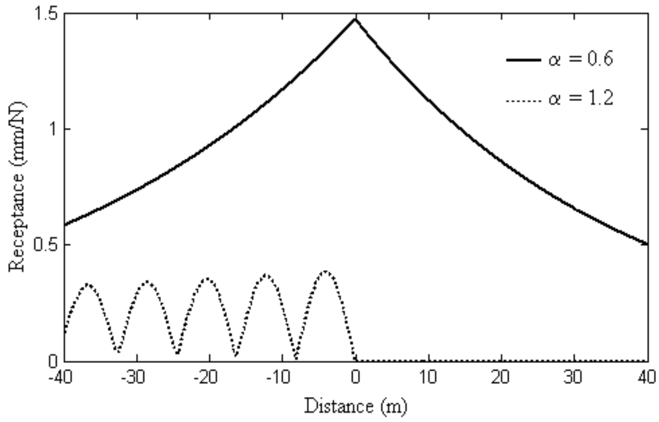

Figure 5. Cross receptance of the wire

Figure 5 presents the cross-receptance of the wire for both sub-critical (α = 0.6) and overcritical velocities (α = 1.2). The angular frequency of the moving harmonic load corresponds to Ω = 0.8. At sub-critical velocity, the elastic waves of the wire are propagating waves, meanwhile at the overcritical velocity, the wire experiences standing waves behind moving load.

CONCLUSIONS

In this paper the dynamic behaviour of an infinite wire on visco-elastic support due to a moving harmonic load is studied. To this end, the Green's functions method has been applied.

There are two cases, the sub-critical one, when the force velocity is lower than the critical velocity (the elastic wave velocity) and the over-critical one, when the velocity of the force is higher than the critical velocity.

The wire response under a stationary harmonic load is similar to the one of the system with a single degree of freedom, excepting the phase resonance. When the harmonic load moves at sub-critical velocities, the resonance frequency decreases and the wire response becomes higher. Also, the propagating waves travel at both ends of the wire. At the overcritical velocities, the wire has only standing waves behind moving harmonic load.

Further research will extend the application of this method (Green's functions method) to the case of the pantograph-catenary interaction.

REFERENCES

[1.] G. Poetsch et al. 1997, Pantograph – catenary dynamics and control, Vehicle System Dynamics 28 159-195.

[2.] G. Bucca, A. Collina, 2009, A procedure for the wear prediction of collector strip and contact wire in pantograph-catenary system, Wear 266, 46–59.

[3.] T. X. Wu, M. J. Brennan 1998, Basic Analytical Study of Pantograph-catenary System Dynamics, Vehicle System Dynamics, 30 443-456.

[4.] J.-W. Kim et al. 2007, State sensitivity analysis of the pantograph system for a high-speed rail vehicle considering span length and static uplift force, Journal of Sound and Vibration 303, 405-427.

[5.] O. Lopez-Garcia, A. Carnicero, J.L. Marono, 2007. Influence of stiffness and contact modelling on catenary-pantograph system dynamics, Journal of Sound and Vibration 303, 405-427.

[6.] L. Fryba, 1972, Vibration of Solids and Structures under Moving Loads, Noordhoff International, Groningen.

[7.] A. V. Metrikine, 2004, Steady state response of an infinite string on a non-linear visco-elastic foundation to moving point loads, Journal of Sound and Vibration 272, 1033-1046.

[8.] A.V. Metrikine, A. L. Bosch, 2006, Dynamic response of a two-level catenary to a moving load, Journal of Sound and Vibration 292, 676-693.

[9.] T. Mazilu, 2010, Application of Green's functions in analysis of the response of an infinite homogenous structure to moving load, U.P.B. Scientific Bulletin, Series A, 72, 139-152.

[10.] T. Mazilu, 2010, New method for calculating response of a periodic structure due to moving loads, Proceedings of the Romanian Academy, Series A 11, 340-346.

[11.] T. Mazilu, 2008, Wheel – Rail Vibrations (in Romanian), Matrixrom, Bucharest.

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