FASTER CALCULATION METHOD FOR UNSTEADY FLOW IN TUBE

1-3. University of Miskolc, Department of Fluid and Heat Engineering, HUNGARY

Abstract: Several methods are known for the calculation of unsteady flow in long tubes having a small-diameter. In case of long pipes having a small-diameter radial change of status indicators are neglected, we consider only the tube longitudinal changes. Most of the calculation methods are based on the finite difference method or the method of equal scale interval characteristic. The common feature of these methods is that the condition for their stability is the fulfilment of the Courant-Friedrich-Lewy condition. This paper shows a faster method for calculation unsteady flow in tube. The governing equations are reduced to three first-order quasi-linear ordinary differential equations. They are solved on the time scale interval analytically. The quickness of this method is given by the used stability condition.

Keywords: unsteady flow, CFD, stability condition

INTRODUCTION
Several methods are known for the calculation of unsteady flow in long tubes having a small-diameter. In case of long pipes having a small-diameter radial change of status indicators are neglected, we consider only the tube longitudinal changes [1]. Most of the calculation methods are based on the finite difference method or the method of equal scale interval characteristic. The common feature of these methods is that the condition for their stability is the fulfilment of the Courant-Friedrich-Lewy condition. This means that for a given spacing step the time step has to fulfil the following equation:

\[ \Delta t \leq \frac{\Delta x}{\max(a+w)}, \]

i.e. the time scales must be less than or equal to the spacing scale divided by the maximum of the sum of the speed of sound and speed of flow. In this paper we show the correlations for frictionless flow in horizontal tube.

THE GOVERNING EQUATIONS
The continuity equation:

\[ \frac{dP}{dt} + \rho \frac{\partial w}{\partial x} = 0. \]

The equation of motion:

\[ \frac{dw}{dt} + \frac{1}{\rho} \frac{\partial P}{\partial x} = 0. \]

Energy equation (Thermodynamics I.):

\[ \frac{dH}{dt} + \frac{1}{\rho} \frac{dP}{dt} = 4k \frac{d(T_1 - T)}{dP}. \]

Thermodynamic properties of fluid:

\[ P = p(\rho, T), \]

\[ h = h(\rho, P). \]

Based on (6) can be written.

\[ \frac{dH}{dt} = \frac{\partial H}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial H}{\partial T} \frac{dT}{dt} \]

Expressed in equation (2)

\[ \frac{1}{\rho} \frac{dP}{dt} = - \frac{\partial w}{\partial x}. \]

Using the correlation

\[ \frac{1}{\rho} \frac{dP}{dt} = - \frac{\partial w}{\partial x} \]

can be written.

Using (7) in the equation (4) it can be written as follows:

\[ \left( \frac{\partial h}{\partial \rho} - \frac{1}{\rho} \right) \frac{dP}{dt} + \frac{\partial h}{\partial T} \frac{dT}{dt} = 4k \left( T_1 - T \right), \]

after rearranging it we get

\[ \left( \frac{\partial h}{\partial \rho} - \frac{1}{\rho} \right) \frac{dP}{dt} + \frac{dP}{dt} = 4k \left( T_1 - T \right) \frac{\partial h}{\partial T}. \]

Using the correlation

\[ \frac{1}{\rho} \frac{dP}{dt} = - \frac{\partial w}{\partial x} \]

can be written.

Expressed in equation (2)

\[ \frac{dP}{dt} = - \frac{\partial w}{\partial x}, \]

and substituting it into equation (11) we get

\[ \frac{dH}{dt} = - \frac{\partial w}{\partial x}. \]
Taking the equation (14) and adding it we get \( a \rho \cdot \) times the equation (3), i.e.

\[
-a^2 \frac{\partial w}{\partial x} \frac{\partial p}{\partial t} - w \frac{\partial p}{\partial x} = b_1.
\]

(14)

and when expressed we get the total derivative of \( u \) by \( t \):

\[
\frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dx}.
\]

(29)

\[ \text{MATHEMATICAL BACKGROUND} \]

Let's consider the following partial differential equation [2], where \( u = u(x,t) \) and where \( c \) and \( k \) are constants:

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} = k\rho,
\]

(25)

Let's take \( c > 0 \), and make the following initial and boundary conditions known.

Initial condition is the following if \( x > 0 \)

\[
u(0,x) = f(x),
\]

(26)

and boundary condition is at \( x = 0 \)

\[
u(0,t) = g(t).
\]

(27)

Let's formulate the total differential of function \( u \):

\[
\frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dx}.
\]

(28)

Comparing the left-hand side of equation (25) and the right side of equation (29) we can write

\[
\frac{du}{dt} = k,
\]

(30)

and

\[
\frac{dx}{dt} = c.
\]

(31)

Solving the ordinary differential equations (30)&(31), the solution is

\[
u(t,x) = kt + f(x),
\]

(32)

and

\[ x = ct + x_0, \]

(33)

where \( x_0 \) is the location coordinate in the \( t = 0 \) moment.

Based on initial condition (26) the value of \( u \) in the \( t = 0 \) moment is:

\[
u(x,0) = f(x_0).
\]

(34)

So the solution of the initial value problem is

\[
u(x,t) = kt + f(x_0),
\]

(35)

When \( x_0 \geq 0 \).

Denoted \( x_0 \) from (33) formula

\[
u(x,t) = kt + f(x - ct),
\]

(36)

If \( f(x_0) = x - ct < 0 \), then the solution is calculated from the boundary condition according to following (Figure 1) [3]:

\[
u(x,t) = \frac{x}{c} + g\left( t - \frac{x}{c} \right).
\]

(37)
If \( c < 0 \), then at location \( x = L \) we specify the boundary condition, i.e. \( u(L, t) = g(t) \). (38)

In this case the solution is the following:

\[ u(x, t) = kt + f(x_0) = kt + f(x - ct), \]

where \( x_0 = x - ct \leq L \), and

\[ u(x, t) = \frac{x - L}{c} + g\left(t - \frac{x - L}{c}\right), \]

when \( x_0 = x - ct > L \).

If the \( c \) is constant, the intersection of characteristics is not possible. If the \( c = c(x, t) \) is a function, then differential equation (31) has only one solution for the given \( [0, t] \) time interval (i.e. the characteristics do not intersect each other [4]), only if the function \( c = c(x, t) \) fulfils the Lipschitz condition. The Lipschitz condition is as follows [5]:

\[ \frac{\partial c}{\partial x} \leq L_c \cdot t \]

Expressing \( c \) from equation (33) we get

\[ c = \frac{x - x_0}{t} \cdot \frac{\partial c}{\partial x}. \]

Using this the Lipschitz condition reformulates as follows:

\[ \left| \frac{x_2 - x_0}{t} \right| - \left| \frac{x_1 - x_0}{t} \right| \leq L_c |x_2 - x_1| \]

and rearranging it we get

\[ L_c \geq \left| \frac{1}{t} \right|. \]

Comparing the equations (42) and (45) it can be written that the function \( c \) fulfills the Lipschitz condition when

\[ \left| \frac{\partial c}{\partial x} \right| t < 1 \]

Introducing the following notation:

\[ a_1 \rho - p = u_1, \]
\[ w_1 = c_1, \]
\[ b_1 = k_1, \]
\[ a_1 \rho_1 w + p = u_2, \]
\[ w_1 + a_1 = c_2, \]
\[ a_1 \rho_1 w - p = u_3, \]
\[ w_1 - a_1 = c_3. \]

Based on them system of equations (24) can be written as follows:

\[ \frac{\partial u_1}{\partial t} + c_1 \frac{\partial u_1}{\partial x} = k_1, \]
\[ \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_1}{\partial x} = -k_1, \]
\[ \frac{\partial u_1}{\partial t} + c_3 \frac{\partial u_1}{\partial x} = k_1. \]

Thus the system is falling into three partial differential equations and according to above their solutions are the followings:

\[ u_1(x, t) = \left\{ \begin{array}{ll}
kt + f(x_0) & \text{if } x \leq L + c_1 t \\
\frac{k x}{c_1} + g_{1,0}\left(t - \frac{x}{c_1}\right) & \text{if } x < c_1 t \\
\frac{k x - L}{c_1} + g_{1,1}\left(t - \frac{x - L}{c_1}\right) & \text{if } x > L + c_1 t
\end{array} \right. \]

\[ u_2(x, t) = \left\{ \begin{array}{ll}
-k t + f(x_0) & \text{if } x \geq c_2 t \\
-k \frac{x}{c_2} + g_{2,0}\left(t - \frac{x}{c_2}\right) & \text{if } x < c_2 t \\
kx + g_{2,1}\left(t - \frac{x}{c_2}\right) & \text{if } x \leq L + c_2 t
\end{array} \right. \]

\[ u_3(x, t) = \left\{ \begin{array}{ll}
\frac{k x - L}{c_3} + g_{3,1}\left(t - \frac{x - L}{c_3}\right) & \text{if } x > L + c_3 t
\end{array} \right. \]

Adding together the equations (50) and (52) and arranging it we get

\[ w = \frac{u_2 + u_3}{a_1 \rho_1 + a_1 \rho_3}. \]

Subtracting equation (52) from equation (50), arranging it and using relation (60) we get

\[ p = \frac{a_1 \rho_1 u_1 - a_1 \rho_3 u_3}{a_1 \rho_2 + a_1 \rho_3}. \]

In terms of the equation (47) and by using relation (61) we get the following formula for density:

\[ p = \frac{1}{a_1^2} \left( u_1 + a_1 \rho_1 u_1 - a_1 \rho_3 u_3 \right) = \frac{a_1 \rho_2 (u_1 + u_3) + a_1 \rho_3 (u_1 - u_3)}{a_1 (a_1 \rho_2 + a_1 \rho_3)}. \]

The uniqueness of the solution is ensured by the fulfillment of the Lipschitz condition. In this case equation (46) is the following:

\[ \max \left| \frac{\partial c}{\partial x} \right| \Delta t < 1. \]
Switching over from differentials to differences can be written that
\[
\Delta t < \frac{\Delta x}{\max(\left|\frac{\partial c}{\partial t}\right|, \left|\frac{\partial c}{\partial x}\right|, \left|\frac{\partial c}{\partial x^2}\right|)}
\]  
(64)

must be met.

**Figure 2.** Symbols of characteristics

**BOUNDARY CONDITIONS**

If point \( M \) is at the inlet of pipe and inflow is here, only the characteristic line from point 3 exists (Figure 3), and according to these and based on the equation (59) the relation between speed and pressure at point \( M \) must be able to meet the following,:
\[
\rho = \rho_3 + a_3 \rho_3 \left( w - w_3 \right) - k \Delta t.
\]  
(65)

This means that if the speed and density are given, then the pressure can be calculated or if the pressure and density are given, the speed can be calculated.

**Figure 3.** Boundary conditions

If point \( M \) is at the end of the pipe and outflow is here, only the characteristics that depart from points 1 and 2 exist. If here the speed is given, then according to (58) the pressure- and according to (57) the density can be calculated, namely as follows:
\[
\rho = \rho_1 + \frac{1}{a_1^2} \left( p - p_1 \right) + k \Delta t.
\]  
(66)

\[
\rho = \rho_1 + \frac{1}{a_1^2} \left( p - p_1 \right) + k \Delta t.
\]  
(67)

It is taken as a special case when inflow is not at the inlet of pipe. It means that \( w_{0,i} = 0 \) (Figure 4.).

**Figure 4.** Boundary conditions without inflow

In this case the characteristic that depart from point 1 also exists at the inlet of pipe. Thus, the properties of point \( M \) that is at inlet of pipe can be computed as follows:
\[
w = 0,
\]  
(68)

\[
\rho = \rho_3 - a_3 \rho_3 \left( w - w_3 \right) - k \Delta t,
\]  
(69)

\[
\rho = \rho_1 + \frac{1}{a_1^2} \left( p - p_1 \right) + k \Delta t.
\]  
(70)

**CONCLUSIONS**

The essence of the method presented here is that the system of equations which describes the flow is reduced to three first-order quasi-linear partial differential equations, which are solved on the \( \Delta t \) time interval where the coefficients of equations are calculated from the status indicators that are known at the beginning of the time interval. The quickness of this method is given by the used stability condition. Here Lipschitz condition (64) must be used instead of Courant-Friedrichs-Lewy condition (1). This means that the calculated time scale for the fixed space scale is not related to the absolute value of the speed of sound and that of the flow speed only their rate of change. In a particular case \( a_{\text{max}} = 391.8 \text{m/s}; w_{\text{max}} = 40.3 \text{m/s} \) of Figure 5 shows the time function of the number of calculation step.

**Figure 5.** Comparison of methods

**ACKNOWLEDGEMENTS**

This research was carried out in the framework of the Centre of Excellence of Innovative Engineering Design and Technologies at the University of Miskolc.

**REFERENCES**


