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HIEMENZ MAGNETIC FLOW BY DIFFERENTIAL TRANSFORMATION METHOD AND PADE APPROXIMANT

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Abstract: The Magnetohydrodynamic (MHD) Hiemenz boundary layer flow over a flat plate embedded in a porous medium in the presence of transverse magnetic field has been studied. The governing equations are solved by differential transformation method with Pade approximant (DTM-Pade) and Runge-Kutta method along with shooting technique. The results of these two methods are compared with the results obtained by finite difference method in conjunction with quasilinearization technique reported earlier in case of the flow without porous medium. It is found that the results of DTM-Pade, Runge-Kutta and quasilinearization technique agree with each other within a certain degree of accuracy. The convergence of the method in attaining the ambient state is faster in case of Runge-Kutta method than the DTM-Pade which can be improved by employing higher dimension Pade approximant matrices. It is also remarked that both magnetic field and porous matrix enhance the velocity field as well as skin friction.
Keywords: Hiemenz magnetic flow; Porous medium; DTM; Finite difference; Quasilinearization

INTRODUCTION

Flows in which the velocity of the incoming fluid is perpendicular to a plane surface is known as Hiemenz flow [1]. If in addition the fluid is electrically conducting, the flow is then called Hiemenz magnetic flow. The solution of this problem is of interest because it is one of the few exact solutions of Navier-Stokes equation in magnetohydrodynamics. Further, the governing equations of the Hiemenz magnetic flow are non-linear. An effective method of solution is the method of finite difference in conjunction with quasilinearization as presented in NA [2].

Applied Mathematics, Physics and problems related to engineering exhibit nonlinear phenomena. Most of nonlinear equations do not have a precise analytical solution; so numerical methods are usually applied to solve the governing equations. Some of the analytical methods are perturbation techniques [3], Adomian decomposition method (ADM) (Dehghan [4-5]), homotopy analysis method (HAM), DTM and variational iteration method (VIM). He [6] and Rashidi [7] have studied the generalized differential transformation method to solve differential equations governing flow of fluids.

The MHD flow finds numerous applications in industries such as MHD power generation and MHD pumps (Hayat et al. ([8]) etc. Further, in the field of heat transfer, the concept of flow through porous media is of great consequence in the modern technology as the porous matrix acts as a good insulator to prevent energy loss. The two relevant properties associated with the study of flow through porous media are porosity and permeability. Porosity basically describes the fraction of total volume which is occupied by the holes. Permeability is a measure of the capacity with which fluids will flow through a porous material. Table 1 presents the numerical values of effective porosity and permeability of materials of common use.

Table 1: Porosity and permeability of typical porous materials

Material	Effective porosity	Permeability
Brick	$0.12-0.34$	$4.8\times10^{-11}-2.2\times10^{-9}$
Copper powder	$0.09-0.34$	$3.3\times10^{-6}-1.5\times10^{-5}$
Leather	$0.56-0.59$	$9.5\times10^{-10}-1.2\times10^{-9}$
Limestone	$0.04-0.10$	$2.0\times10^{-11}-4.5\times10^{-10}$
Sand	$0.37-0.50$	$2.0\times10^{-7}-1.8\times10^{-6}$
Sand stone	$0.08-0.38$	$5.0\times10^{-12}-3.0\times10^{-8}$
Silica powder	$0.37-0.49$	$1.3\times10^{-10}-5.1\times10^{-10}$
Soil	$0.43-0.54$	$2.9\times10^{-9}-1.4\times10^{-7}$
Wire crimps	$0.68-0.76$	$3.8\times10^{-5}-1.0\times10^{-4}$

The objective of the present study is to apply DTM, DTM-Pade approximant and Runge-Kutta method to solve modified Navier-Stokes equation for Hiemenz magnetic flow through porous media and to compare the results obtained by the present methods of solution with the results reported in [2] employing finite difference method in conjunction with quasilinearization technique.

HIEMENZ MAGNETIC FLOW

The boundary layer equations for Hiemenz magnetic Darcy flows for viscous fluid following [9] are:

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

Momentum:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=a^2x+\upsilon\frac{\partial^2 u}{\partial y^2}+\frac{\sigma B^2}{\rho}(ax-u)+\frac{\upsilon}{K_p^*}(ax-u) \qquad (2)$$

subject to the boundary conditions:

$$\left.\begin{aligned} &u=0,\ v=0 \quad at\ \ y=0\\ &u=ax, \qquad at\ \ y\to\infty \end{aligned}\right\} \qquad (3)$$

where $u, v, \upsilon, \rho, K_p^*$ and a are respectively the x-component of the velocity, the y-component of the velocity, the viscosity, the density, the permeability of the medium and a constant known as initial stretching rate (characteristic of the incoming flow) with dimension (time)$^{-1}$. Further, σ and B are respectively the electrical conductivity and magnetic induction. The last term represents the additional resistance due to porosity of the porous medium. The flow geometry is shown in Figure 1.

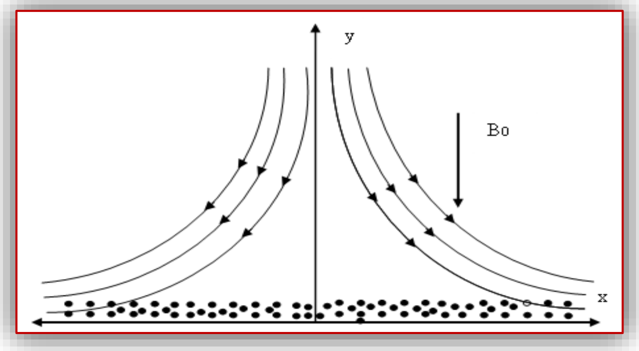

Figure 1. Flow geometry

The first two prescribed boundary conditions represent neither slip nor mass transfer on the surface where the conditions are at infinity (i.e. ambient state) means that the velocity of the fluid approaches a linear relation with x.

Introducing the variables

$$u=ax\frac{df}{d\eta} \ \text{ and } \ v=-\sqrt{a\upsilon}\,f(\eta), \qquad (4)$$

where $$\eta=\sqrt{\frac{a}{\upsilon}}\,y. \qquad (5)$$

the equations (2) and (3) become

$$\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}+1-\left(\frac{df}{d\eta}\right)^2+\left(M+\frac{1}{K_p}\right)\left(1-\frac{df}{d\eta}\right)=0 \qquad (6)$$

$$\left.\begin{aligned} &f=0,\ \frac{df}{d\eta}=0 \ \ at\,\eta=0\\ &\frac{df}{d\eta}=1 \qquad\quad at\,\eta\to\infty \end{aligned}\right\} \qquad (7)$$

where $M=\dfrac{\sigma B^2}{a\rho}$ is the magnetic parameter and $K_p=\dfrac{aK_p^*}{\upsilon}$ is the dimensionless permeability parameter.

Case–I: $M=0, K_p\to\infty$ (for large value of K_p), the problem reduces to the flow of fluids without magnetic field and porous medium.

Case-II: $M=0$ and K_p finite (for small value of K_p), the problem represents Newtonian flow through porous medium without magnetic field.

Case-III: $M\neq 0$ and K_p finite (for small value of K_p), the problem represents the Darcy flow of conducting fluid in the presence of magnetic field with low magnetic parameter.

Solution of equation (6) with boundary conditions (7) is obtained by employing differential transformation method with Pade approximant (DTM-Pade) and Runge-Kutta method. The results of these two methods are compared with the results obtained by finite difference method in conjunction with quasilinearization [2].

DIFFERENTIAL TRANSFORMATION METHOD

Differential transformation method is a numerical method based on Taylor's expansion. This method determines the coefficients of series expansion of unknown function by using the initial data on the problem. The concept of differential transformation method was first proposed by Zhou [10]. The DTM-Pade was applied to electric circuit analysis problems and also it was applied to several systems of differential equations for example, initial value problems [11], difference equations [12], integro-differential equations [13], and partial differential equations [14].

Definition 1. The one dimensional differential transform of a function $f(\eta)$ at the point $\eta=\eta_0$ is defined as

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{d\eta^k}\{f(\eta)\}\right]_{\eta=\eta_0} \qquad (8)$$

where $f(\eta)$ is the original function and $F(k)$ is the transformed function.

Definition 2. The differential inverse transform of $F(k)$ is defined as

$$f(\eta)=\sum_{k=0}^{\infty}F(k)(\eta-\eta_0)^k \qquad (9)$$

The other properties are enlisted in Table 2.

Table 2: Some properties of differential transformation method.

Original function	Transformed function
$f(\eta)=f_1(\eta)\pm f_2(\eta)$	$F(k)=F_1(k)\pm F_2(k)$
$f(\eta)=\lambda f_1(\eta)$	$F(k)=\lambda F_1(k)$
$f(\eta)=f_1(\eta)f_2(\eta)$	$F(k)=\sum_{r=0}^{k}F_1(r)F_2(k-r)$
$f(\eta)=\dfrac{d^n f_1(\eta)}{d\eta^n}$	$F(k)=\dfrac{(k+n)!}{k!}F_1(k+n)$
$f(\eta)=f_1(\eta)\dfrac{d^2 f_2(\eta)}{d\eta^2}$	$F(k)=\sum_{r=0}^{k}(k-r+1)(k-r+2)F_1(r)F_2(k-r+2)$
$f(\eta)=(\eta-\eta_0)^m$	$F(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}$

PADE APPROXIMANT

Some techniques exist to accelerate the convergence of a given series. Among them the so-called Pade approximant is widely applied (Baker and Morris, [15]). Suppose that a function $f(\eta)$ is represented by a power series,

$$f(\eta)=\sum_{i=0}^{\infty}c_i\eta^i \qquad (10)$$

This expression is the fundamental point of any analysis using Pade approximant. The notation $c_i, i=0,1,2\cdots$ is reserved for the given set of coefficients and $f(\eta)$ is the associated function. $[L/M]$ Pade approximant is a rational fraction, defined as

$$f(\eta)=\frac{a_0+a_1\eta+a_2\eta^2+\cdots+a_L\eta^L}{b_0+b_1\eta+b_2\eta^2+\cdots+b_M\eta^M}, \qquad (11)$$

which has a Maclaurin expansion, agrees with equation (10) as far as possible. It is noticed that in equation (11) there are L+1 numerator and M+1 denominator coefficients. So there are L+1 independent numerator and M independent denominator coefficients, making L+M+1 unknown coefficients in all. This number suggests that normally $[L/M]$ ought to fit the power series equation (10) through the orders $1,\eta,\eta^2\cdots\eta^{L+M}$. In the notation of formal power series

$$\sum_{i=0}^{\infty}c_i\eta^i=\frac{a_0+a_1\eta+a_2\eta^2+\cdots+a_L\eta^L}{b_0+b_1\eta+b_2\eta^2+\cdots+b_M\eta^M}+O(\eta^{L+M+1}) \qquad (12)$$

$$\left(b_0+b_1\eta+\cdots+b_M\eta^M\right)\left(c_0+c_1\eta+\cdots\right)=a_0+a_1\eta+\cdots+a_L\eta^L+O(\eta^{L+M+1}) \qquad (13)$$

Equating the coefficients of $\eta^{L+1},\eta^{L+2},\cdots,\eta^{L+m}$ we get,

$$\left.\begin{aligned}
&b_M c_{L-M+1}+b_{M-1}c_{L-M+2}+\cdots+b_1c_L+b_0c_{L+1}=0,\\
&b_M c_{L-M+2}+b_{M-1}c_{L-M+3}+\cdots+b_1c_{L+1}+b_0c_{L+2}=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&b_M c_L+b_{M-1}c_{L+1}+\cdots+b_1c_{L+M-1}+b_0c_{L+M}=0,
\end{aligned}\right\} \qquad (14)$$

If $j<0$, we define $c_i=0$ for consistency. Since $b_0=1$, equation (14) becomes a set of M linear equations for M unknown denominator coefficients.

$$\begin{pmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L\\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1}\\ \vdots & \vdots & \vdots & \vdots\\ c_L & c_{L+1} & \cdots & c_{L+M-1}\end{pmatrix}\begin{pmatrix} b_M\\ b_{M-1}\\ \vdots\\ b_1\end{pmatrix}=-\begin{pmatrix} c_{L+1}\\ c_{L+2}\\ \vdots\\ c_{L+M}\end{pmatrix} \qquad (15)$$

From these equations, b_i may be found. The numerator coefficients $a_0,a_1,\cdots,a_L$, follow immediately from equation (13) by equating the coefficients of $1,\eta,\eta^2,\cdots,\eta^{L+M}$ such as,

$$\left.\begin{aligned}
&a_0=c_0,\\
&a_1=c_1+b_1c_0,\\
&a_2=c_2+b_1c_1+b_2c_0,\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\\
&a_L=c_L+\sum_{i=1}^{\min[L/M]}b_ic_{L-i.}
\end{aligned}\right\} \qquad (16)$$

Thus equations (15) and (16) normally determine the Pade numerator and denominator and are called Pade equations. The [L/M] Pade approximant is constructed which agrees with the equation (12) through the order η^{L+M}.

SOLUTION OF THE PROBLEM

Analytical solution

Consider the equation (4)

$$f'''(\eta)+f(\eta)f''(\eta)+1-\left(f'(\eta)\right)^2+\left(M+\frac{1}{K_p}\right)\left(1-f'(\eta)\right)=0, \qquad (17)$$

with boundary conditions

$$f(0)=0,\ f'(0)=0,\ f'(\infty)=1 \qquad (18)$$

Combination of the series obtained by DTM and Pade approximant will yield the numerical value of $f''(0)$ so as to reduce the present boundary value problem (BVP) into an initial value problem (IVP). The diagonal Pade approximants of degree $[2/2]$ is employed to determine the approximate solution.

Let $f''(0) = 2A$, where A is a positive constant. Now, the differential transform method (DTM) will be applied to equation (17) as follows:

$$(k+1)(k+2)(k+3)F(k+3)$$
$$+\sum_{r=0}^{k}\{(k-r+1)(k-r+2)F(r)F(k-r+2)$$
$$-(r+1)(k-r+1)F(r+1)F(k-r+1)\}$$
$$+\left(M+\frac{1}{K_p}+1\right)\delta(k)-\left(M+\frac{1}{K_p}\right)(k+1)F(k+1)=0 \quad (19)$$

The differential transform of boundary conditions are

$$F(0)=0, F(1)=0, F(2)=A. \quad (20)$$

Applying the differential inverse transform,

$$f(\eta)=\sum_{k=0}^{\infty}F(k)\eta^k$$
$$=A\eta^2-\frac{\left(M+\frac{1}{K_p}+1\right)}{6}\eta^3+\frac{\left(M+\frac{1}{K_p}\right)A}{12}\eta^4$$
$$+\left[\frac{A^2}{30}-\frac{\left(M+\frac{1}{K_p}\right)\left(M+\frac{1}{K_p}+1\right)}{120}\right]\eta^5+\left[\frac{\left(M+\frac{1}{K_p}\right)^2 A}{160}-\frac{\left(M+\frac{1}{K_p}+1\right)A}{180}\right]\eta^6\cdots \quad (21)$$

Case I : ($M=0.5$, $K_p=100$)

The DTM expression (21) becomes

$$f(\eta)=A\eta^2-\frac{1}{4}\eta^3+\frac{A}{24}\eta^4+\left(\frac{A^2}{30}-\frac{1}{160}\right)\eta^5+\frac{11A}{20160}\eta^6+\left(-\frac{A}{21504}+\frac{A^3}{161280}\right)\eta^7$$
$$-\frac{43}{967680}\eta^8+\left(\frac{A}{552960}-\frac{5A^3}{387072}\right)\eta^9+\cdots \quad (22)$$

Now our aim is to determine A using the boundary condition

$$\lim_{\eta\to-\infty} f'(\eta)=1 \quad (23)$$

Applying the boundary condition (23) to [2/2] Pade approximant of the derivative of the polynomial solution (22), we get

$$\lim_{\eta\to\infty}\frac{2A\eta+\left(\frac{64A^4+72A^2-81}{27-16A^2}\right)\eta^2}{1-\left(\frac{3A+8A^3}{27-16A^2}\right)\eta-\left[\frac{27+112A^2}{24(27-16A^2)}\right]\eta^2}=1$$

which gives $A=0.8184854107$.

Table 3: Determination of A

M	K_p	A
0.5	100	0.8184854107
1	100	0.8660254038
2	100	0.8918112327
0.5	0.5	0.8011621274
1	0.5	0.9632417272
2	0.5	1.0032411257

Similarly, the following values of A are obtained for the various values of M and K_p as depicted in Table 3.

NUMERICAL SOLUTION

The governing equation is solved numerically by applying fourth order Runge-Kutta method along with shooting technique. This method has been proven to be adequate and gives accurate results for boundary layer equation. The solution is computed for the dimensionless velocity and shown graphically.

Let $f(\eta)=y_1, f'(\eta)=y_2$ and $f''(\eta)=y_3$.

so that

$$y'_3=-y_1y_3-1+y_2^2-\left(M+\frac{1}{K_p}\right)(1-y_2)$$

with $y_a(1)=0,\ y_a(2)=0, y_b(2)=1$.

RESULTS AND DISCUSSION

The DTM-Pade approximant and Runge-Kutta method with shooting technique have been applied for solving Hiemenz magnetic flow through porous medium. The solution for the flow without porous medium has been derived as a particular case and the results are compared with the results obtained by the method of finite difference in conjunction with quasilinearization technique in Na [2].

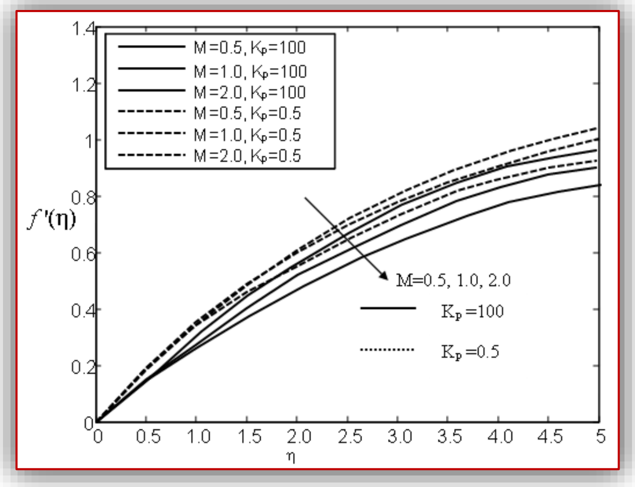

Figure 2. Velocity profiles (DTM)

Figures 2, 3 and 4 exhibit the results obtained by DTM, DTM-Pade and Runge-Kutta method associated with shooting technique. It is found that the effect of magnetic parameter is to decrease the velocity of the fluid irrespective of the presence or absence of porous matrix. On careful observation it is further remarked that presence of porous matrix also decreases the velocity profiles at all the points. Therefore, it is concluded that presence of porous matrix as well as magnetic field both decrease the velocity of the fluid at all points of the flow domain. Further, it is seen that rate of decrease is more with the combined effect of the magnetic field and porous matrix. From figures 3 and 4 it is observed that the attainment of ambient state is faster in case of DTM-Pade in comparison with DTM and it is still faster in case of Runge-Kutta method.

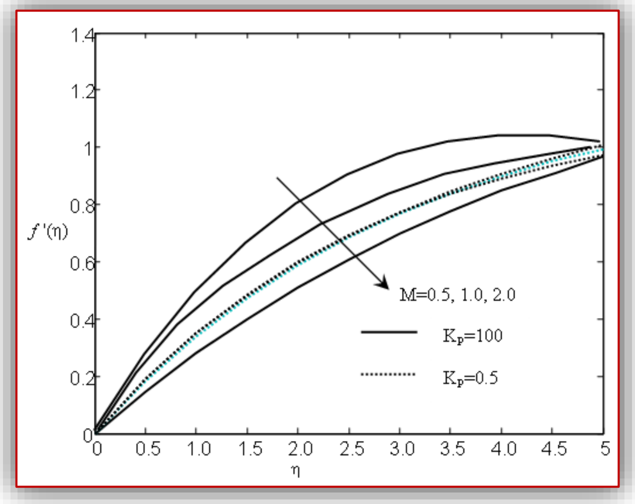

Figure 3. Velocity profiles (DTM-Pade).

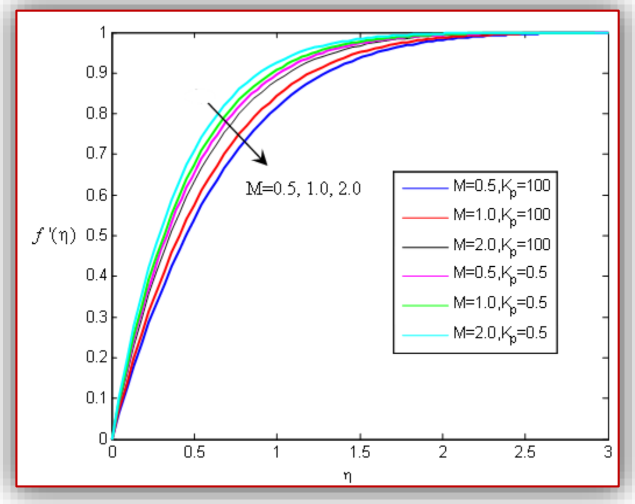

Figure 4. Velocity profiles (Runge-Kutta method).

Table 4: Skin friction coefficient (2A)

M	Kp	DTM-Pade	Runge-Kutta	Quasilinearization NA [2]
0.5	100	1.6368	1.3832	1.362
1	100	1.7320	1.5885	1.5394
2	100	1.7836	1.8761	1.833
0.5	0.5	1.6023	2.0022	-
1	0.5	1.9265	2.1232	-
2	0.5	2.0065	2.3466	-

Now, the Table-4 presents the numerical values of skin friction computed by different methods. It is evident that skin friction (in magnitude) increases with an increase in the values of magnetic parameter with or without porous medium. It is also seen that presence of porous matrix increases the skin friction (in magnitude) for a fixed value of magnetic parameter.

Table-4 further reveals that the values of skin friction obtained by (i) Runge-Kutta and (ii) Finite difference in conjunction with quasilinearization agree up to first decimal place. This shows the consistency of the methods applied in the present analysis to solve the modified MHD Hiemenz flow. It is suggested that the accuracy of DTM-Pade method can be improved by employing higher degree diagonal Pade approximants.

CONCLUSION

The DTM-Pade and Runge-Kutta method are consistent within certain degree of accuracy to solve non-linear boundary value problems and convergence of the method can be accelerated with higher dimension Pade approximant matrices so as to attain the ambient state of the flow which is also assisted by the presence of magnetic field and porous medium. It is further concluded that presence of magnetic field and porous medium is found to be counterproductive in reducing the skin friction at the surface of the plate.

References

[1.] K. Hiemenz, Die Grenzschicht an einem n den gleichformigen Flussigkeitsstrom eingetauchten geraden Kreiszylinder, Dingl. Polytech. J. 326 (1911) 321.
[2.] T.Y. NA, Computational methods in Engineering Boundary Value Problems, Academic press, New York (1979) 120-121.
[3.] J. H. He, Homotopy perturbation technique, Computer Methods in Appl. Mech. Eng., 178 (3-4) (1999) 257–262.
[4.] M. Dehghan, F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic fields using He's homotopy perturbation method, Progress in Electromagnetic Research, 78 (2008) 361–376.
[5.] M. Dehghan, F. Shakeri, Use of He's homotopy perturbation method for solving a partial differential equation arising in modeling of flow in porous media, Journal of Porous Media, 11 (2008) 765–778.
[6.] J. H. He, A new approach to non-linear partial differential equations, Comm. Nonlinear Sci. Numer. Simulation, 2(4) (1997) 230-235.
[7.] M. M. Rashidi, H. Shahmohamadi, Analytical solution of three-dimensional Navier–Stokes equations for the flow near an infinite rotating disk, Comm. Nonlinear Sci. Numer. Simulat., 14 (7) (2009) 2999–3006.
[8.] T. Hayat, C. Fetecau, M. Sajid, Analytic solution for MHD transient rotating flow of a second grade fluid in a porous space, Nonlinear Analysis: Real World Applications, 9 (4) (2008) 1619–1627.

[9.] Dj. S. SjuKic, Hiemenz magnetic flow of power-law fluids, J. Appl. Mech. Trans. ASME, 40 (1974) 822-823.
[10.] J. K. Zhou, Differential transformation and its applications for electrical circuits, Huazhong University Press, Wuhan, China, 1986.
[11.] M. J. Jang, C. L. Chen, Y. C. Liu, On solving the initial value problems using the differential transformation method, Appl. Math. Compt., 115 (2000) 145-160.
[12.] A. Arikoglu, I. Ozkol, Solution of difference equations by using differential transform method, Appl. Math. Compt., 174 (2006) 1216–1228.
[13.] P. Darania, A.Ebadian, A method for the numerical solution of integro-differential equations, Appl. Math. Compt., 188 (2007) 657-668.
[14.] F. Ayaz, On the two-dimensional differential transform method, Appl. Math. Compt., 143 (2-3) (2003) 361-374
[15.] G.A. Baker, P. Graves-Morris, Pade Approximants, Cambridge U.P., 1996

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