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## ANALYSIS OF BIMETALLIC BEAM WITH WEAK SHEAR CONNECTION

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**Abstract:** In this paper an analytical solution is presented to determine the deflection, slip and stresses in bimetallic beam with flexible shear connection. The thermal load is derived from uniform temperature change. The Euler-Bernoulli hypothesis is assumed to hold for each layer separately and a linear constitutive equation between the horizontal slip and the inter-laminar shear force is considered. An example illustrates the application of the developed analytical method.

**Keywords:** bimetallic beam, interlayer slip, shear connection, thermal load

### INTRODUCTION

There exist several works on bimetallic elastic beams with perfect bond [1,2,3,4,5]. In this paper the bimetallic beam with weak shear connection under the action of uniform temperature change is studied. The present analytical method is based on the Euler-Bernoulli's beam theory and the one-dimensional version of the constitutive equation of linear thermoelasticity (Duhamel-Neumann's law). The considered bimetallic beam configuration is shown in Figure 1.

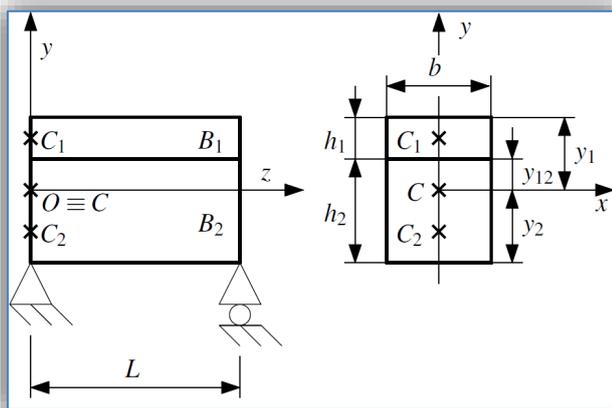


Figure 1. Simply supported bimetallic beam

The beam component  $B_i$  has the rectangular cross section  $A_i$  whose dimensions are  $h_i$  and  $b$  ( $i=1,2$ ). The modulus of elasticity for beam component  $B_i$  is  $E_i$  and the coefficients of linear thermal

expansion is  $\alpha_i$  ( $i=1,2$ ). The length of the simply supported bimetallic beam is  $L$ . The origin  $O$  of the rectangular Cartesian coordinate system  $Oxyz$  is the  $E$ -weighted centre of the left end cross section, so that axis  $z$  is the  $E$ -weighted center-line of the bimetallic beam. A point  $P$  in  $B = B_1 \cup B_2$  is indicated by the position vector  $\overline{OP} = \mathbf{r} = \mathbf{R} + z\mathbf{e}_z = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ , where  $\mathbf{e}_x$ ,  $\mathbf{e}_y$  and  $\mathbf{e}_z$  are the unit vectors of the coordinate system  $Oxyz$ . It is known that the position of  $E$ -weighted centre of the cross section  $A = A_1 \cup A_2$  is obtained from next equation [6]

$$E_1 \int_{A_1} \mathbf{r} dA + E_2 \int_{A_2} \mathbf{r} dA = \mathbf{0}. \quad (1)$$

For cross section shown in Figure 1 we have

$$c_1 = |\overline{CC_1}| = \frac{A_2 E_2}{\langle AE \rangle} c, \quad c_2 = -|\overline{CC_2}| = -\frac{A_1 E_1}{\langle AE \rangle} c, \quad (2)$$

$$c = |\overline{C_2 C_1}| = c_1 - c_2 = \frac{1}{2}(h_1 + h_2), \quad (3)$$

$$\langle AE \rangle = A_1 E_1 + A_2 E_2, \quad (4)$$

$$y_1 = c_1 + \frac{1}{2}h_1, \quad y_2 = c_2 - \frac{1}{2}h_2, \quad y_{12} = c_1 - \frac{1}{2}h_1. \quad (5)$$

In Eqs. (2), (4)  $A_i$  denotes the cross sectional area of beam component  $B_i$  ( $i=1,2$ ) and the position of the common boundary of  $A_1$  and  $A_2$  is indicated by  $y_{12}$  (Figure 1).

### GOVERNING EQUATION

According to the Euler-Bernoulli hypothesis (kinematic assumption) which is valid for each homogeneous beam components the deformed configuration is described by the displacement field [6]

$$\mathbf{u} = \mathbf{u}(x, y, z) = v(z)\mathbf{e}_y + \left( w_i(z) - y \frac{dv}{dz} \right) \mathbf{e}_z, \quad (6)$$

where  $(x, y, z) \in B_i$ ,  $(i=1,2)$ . Eq. (6) shows that the axial displacement of beam component  $B_i$  ( $i=1,2$ ) is separated into two parts:  $w_i(z)$  ( $i=1,2$ ) describes the rigid translation of the cross section  $A_i$  ( $i=1,2$ ) at  $z$  and the second part of the axial displacement of  $A_i$  ( $i=1,2$ ) derived from the deflection of cross section [6]. On the common boundary of  $B_1$  and  $B_2$  the axial displacement has jump which is called the interlayer slip. According to Eq. (6) the interlayer slip  $s = s(z)$  can be computed as

$$s(z) = w_1(z) - w_2(z). \quad (7)$$

Application of the strain-displacement relationships of the linearized theory of elasticity gives

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0, \quad (x, y, z) \in B_1 \cup B_2, \quad (8)$$

$$\varepsilon_z = \frac{dw_i}{dz} - y \frac{d^2v}{dz^2}, \quad (x, y, z) \in B_i, \quad (i=1,2). \quad (9)$$

In Eqs. (8), (9)  $\varepsilon_x$ ,  $\varepsilon_y$ ,  $\varepsilon_z$  are the normal strains and  $\gamma_{xy}$ ,  $\gamma_{xz}$ ,  $\gamma_{yz}$  are the shearing strains. The normal stress  $\sigma_z$  is computed from the one-dimensional version of Duhamel-Neumann's law [3,4]

$$\sigma_z = E_i \left( \frac{dw_i}{dz} - y \frac{d^2v}{dz^2} - \alpha_i T \right), \quad (x, y, z) \in B_1 \cup B_2. \quad (10)$$

In Eq. (10)  $T$  denotes the temperature change. The temperature of the two-layer composite beam initially is the reference temperature. Its temperature is slowly raised to a constant uniform temperature, so that the temperature change is  $T$ . Following we define the next section forces and moments [6]

$$N_1 = \int_{A_1} \sigma_z dA = A_1 E_1 \left( \frac{dw_1}{dz} - c_1 \frac{d^2v}{dz^2} - \alpha_1 T \right), \quad (11)$$

$$N_2 = \int_{A_2} \sigma_z dA = A_2 E_2 \left( \frac{dw_2}{dz} - c_2 \frac{d^2v}{dz^2} - \alpha_2 T \right), \quad (12)$$

$$M_1 = \int_{A_1} y \sigma_z dA = A_1 E_1 c_1 \left( \frac{dw_1}{dz} - \alpha_1 T \right) - E_1 I_1 \frac{d^2v}{dz^2}, \quad (13)$$

$$M_2 = \int_{A_2} y \sigma_z dA = A_2 E_2 c_2 \left( \frac{dw_2}{dz} - \alpha_2 T \right) - E_2 I_2 \frac{d^2v}{dz^2}, \quad (14)$$

where

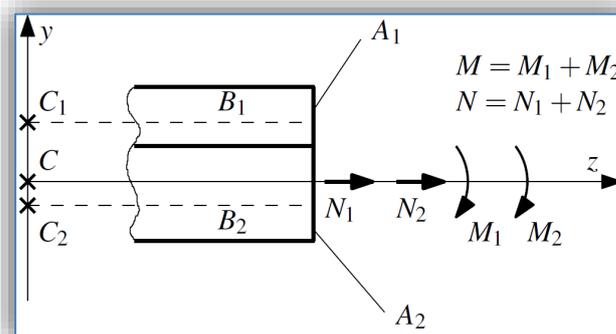
$$I_i = \int_{A_i} y^2 dA, \quad (i=1,2). \quad (15)$$

Eqs. (11), (12), (13) and (14) show that the normal stresses acting on cross section  $A_i$  ( $i=1,2$ ) are equivalent to a force-couple system  $(N_i, M_i)$  ( $i=1,2$ ) at  $C$ . This force-couple system  $(N_i, M_i)$  ( $i=1,2$ ) is illustrated in Figure 2. The interlayer slip  $s$  is assumed to be a linear function of shear force  $Q$  transmitted between the two beam components, that is we have [7]

$$Q = ks, \quad (16)$$

where  $k$  is a constant, it is called slip modulus. Units of  $Q$  and  $k$  are

$$[Q] = \frac{\text{force}}{\text{length}}, \quad [k] = \frac{\text{force}}{(\text{length})^2}. \quad (17)$$



**Figure 2.** Normal forces and bending moments

In present problem there is no axial force  $N = N_1 + N_2$ , that is

$$N = N_1 + N_2 = A_1 E_1 \frac{dw_1}{dz} + A_2 E_2 \frac{dw_2}{dz} - \langle AE\alpha \rangle T = 0. \quad (18)$$

Here,

$$\langle AE\alpha \rangle = A_1 E_1 \alpha_1 + A_2 E_2 \alpha_2. \quad (19)$$

From Eqs. (7) and (18) it follows that

$$\frac{dw_1}{dz} = \frac{A_2 E_2}{\langle AE \rangle} \frac{ds}{dz} + \frac{\langle AE\alpha \rangle}{\langle AE \rangle} T, \quad (20)$$

$$\frac{dw_2}{dz} = -\frac{A_1 E_1}{\langle AE \rangle} \frac{ds}{dz} + \frac{\langle AE\alpha \rangle}{\langle AE \rangle} T. \quad (21)$$

A simple computation based on Eqs. (11), (12) and Eqs. (20), (21) gives

$$N_1 = \langle AE \rangle_{-1} \left[ \frac{ds}{dz} - c \frac{d^2v}{dz^2} + (\alpha_2 - \alpha_1) T \right], \quad (22)$$

$$N_2 = \langle AE \rangle_{-1} \left[ -\frac{ds}{dz} + c \frac{d^2v}{dz^2} + (\alpha_1 - \alpha_2) T \right], \quad (23)$$

where

$$\langle AE \rangle_{-1} = \frac{1}{\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}}. \quad (24)$$

Application of the condition of equilibrium for forces in axial direction to beam component  $B_1$  gives (Figure 3)

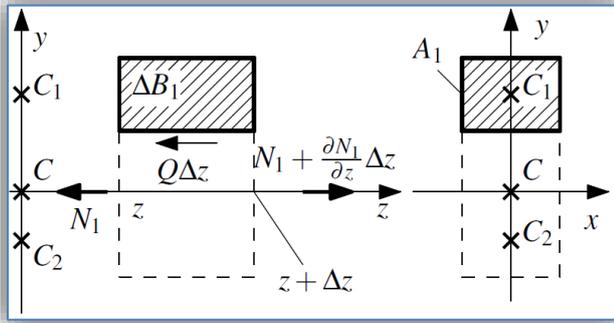


Figure 3. Equilibrium condition in  $z$  direction for a small beam element  $\Delta B_1$

$$\frac{dN_1}{dz} - ks = 0. \quad (25)$$

Substitution of Eq. (22) into Eq. (25) yields

$$\frac{d^2 s}{dz^2} - c \frac{d^3 v}{dz^3} - \frac{k}{\langle AE \rangle_{-1}} s = 0. \quad (26)$$

It is evident the bending moment acting on the whole cross section  $A = A_1 \cup A_2$  is as follows

$$M = M_1 + M_2 = c \langle AE \rangle_{-1} \left[ \frac{ds}{dz} + (\alpha_2 - \alpha_1) T \right] - \{IE\} \frac{d^2 v}{dz^2}. \quad (27)$$

Here,

$$\{IE\} = I_1 E_1 + I_2 E_2. \quad (28)$$

There is no applied mechanical load on the whole two-layer composite beam and at both supports there are not any reaction forces, so that

$$M(z) = 0, \quad V(z) = \frac{dM}{dz} = 0 \quad (29)$$

for all cross section. In Eq. (29)<sub>2</sub>  $V = V(z)$  is the cross-sectional shear force. From Eq. (29)<sub>2</sub> we get

$$\frac{d^3 v}{dz^3} = c \frac{\langle AE \rangle_{-1}}{\{IE\}} \frac{d^2 s}{dz^2}. \quad (30)$$

Combination of Eq. (26) with Eq. (30) gives

$$\frac{d^2 s}{dz^2} - \Omega^2 s = 0, \quad (31)$$

where

$$\Omega^2 = k \frac{\{IE\}}{\langle AE \rangle_{-1} \{IE\}}, \quad \langle IE \rangle = \{IE\} - c^2 \langle AE \rangle_{-1}. \quad (32)$$

### DETERMINATION OF THE SLIP AND DEFLECTION

For the simply supported bimetallic beam shown in Figure 1 the following boundary conditions are valid

$$v(0) = 0, \quad v(L) = 0, \quad (33)$$

$$N_1(0) = 0, \quad N_1(L) = 0. \quad (34)$$

The boundary conditions for bending moment  $M = M(z)$

$$M(0) = 0, \quad M(L) = 0 \quad (35)$$

are satisfied according to Eq. (29). From the boundary conditions

$$N_1(0) = 0, \quad M(0) = 0 \quad (36)$$

and

$$N_1(L) = 0, \quad M(L) = 0 \quad (37)$$

it follows that

$$\frac{ds}{dz} = (\alpha_1 - \alpha_2) T \quad (38)$$

is valid for  $z = 0$  and  $z = L$ . The general solution of the differential equation (31) can be represented as

$$s(z) = K_1 \cosh \Omega z + K_2 \sinh \Omega z. \quad (39)$$

Substitution of Eq. (39) into the boundary condition (38) leads to the next results

$$K_1 = -T \frac{\alpha_1 - \alpha_2}{\Omega} \tanh \frac{\Omega L}{2}, \quad (40)$$

$$K_2 = T \frac{\alpha_1 - \alpha_2}{\Omega}. \quad (41)$$

From Eqs. (27), (29)<sub>1</sub> and Eq. (39) it follows that

$$c \langle AE \rangle_{-1} [K_1 (\cosh \Omega z - 1) + K_2 \sinh \Omega z + (\alpha_2 - \alpha_1) T z] - \{IE\} \frac{dv}{dz} + \{IE\} K_3 = 0, \quad (42)$$

where

$$K_3 = \left( \frac{dv}{dz} \right)_{z=0}. \quad (43)$$

Integrating of Eq. (42) gives

$$\{IE\} [v(z) - v(0)] = c \langle AE \rangle_{-1} \left[ K_1 \frac{\sinh \Omega z - \Omega z}{\Omega} + K_2 \frac{\cosh \Omega z - 1}{\Omega} + \frac{\alpha_2 - \alpha_1}{2} T z^2 \right] + \{IE\} K_3 z. \quad (44)$$

From boundary conditions (33) we obtain

$$K_3 = -\frac{c \langle AE \rangle_{-1}}{\{IE\}} \left[ K_1 \frac{\sinh \Omega L - \Omega L}{\Omega} + K_2 \frac{\cosh \Omega L - 1}{\Omega} + \frac{L}{2} (\alpha_2 - \alpha_1) T \right]. \quad (45)$$

Substitution of Eq. (45) into Eq. (44) gives

$$v(z) = \frac{c \langle AE \rangle_{-1}}{\{IE\}} \left[ K_1 \left( \frac{\sinh \Omega z - \Omega z}{\Omega} - \frac{\sinh \Omega L - \Omega L}{\Omega L} z \right) + K_2 \left( \frac{\cosh \Omega z - 1}{\Omega} - \frac{\cosh \Omega L - 1}{\Omega L} z \right) - \frac{\alpha_2 - \alpha_1}{2} T (Lz - z^2) \right]. \quad (46)$$

**COMPUTATIONS OF THERMAL STRESSES**

We assume that the state of stresses of bimetallic beam can be characterized by the following stresses  $\sigma_z = \sigma_z(y, z)$ ,  $\tau_{yz} = \tau_{yz}(y, z)$ ,  $\sigma_y = \sigma_y(y, z)$ . The normal stress  $\sigma_z$  is obtained from Eqs. (10) and (20) as

$$\sigma_z = E_1 \left[ \frac{c_1}{c} \frac{ds}{dz} - y \frac{d^2v}{dz^2} + \frac{c_1}{c} (\alpha_2 - \alpha_1) T \right], \quad (x, y, z) \in B_1, \quad (47)$$

$$\sigma_z = E_2 \left[ \frac{c_2}{c} \frac{ds}{dz} - y \frac{d^2v}{dz^2} + \frac{c_2}{c} (\alpha_2 - \alpha_1) T \right], \quad (x, y, z) \in B_2. \quad (48)$$

Shearing stresses  $\tau_{yz} = \tau_{yz}(y, z)$  is computed by the use of equation of equilibrium

$$\frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0, \quad (x, y, z) \in B_1 \cup B_2. \quad (49)$$

A detailed computation yields the next result

$$\tau_{yz} = -E_2 \left[ (y - y_2) \frac{c_2}{c} \frac{d^2s}{dz^2} - \frac{1}{2} (y^2 - y_2^2) \frac{d^3v}{dz^3} \right], \quad (x, y, z) \in B_2, \quad (50)$$

$$\begin{aligned} \tau_{yz} = & -E_2 \left[ (y_{12} - y_2) \frac{c_2}{c} \frac{d^2s}{dz^2} - \frac{1}{2} (y_{12}^2 - y_2^2) \frac{d^3v}{dz^3} \right] - \\ & -E_1 \left[ (y - y_{12}) \frac{c_1}{c} \frac{d^2s}{dz^2} - \frac{1}{2} (y^2 - y_{12}^2) \frac{d^3v}{dz^3} \right], \quad (51) \\ & (x, y, z) \in B_1. \end{aligned}$$

Here, the stress boundary condition

$$\tau_{yz}(y_2, z) = 0 \quad (52)$$

and the continuity condition of  $\tau_{yz}$  at  $y = y_{12}$

$$\lim_{\varepsilon \rightarrow 0} [\tau_{yz}(y_{12} - \varepsilon, z) - \tau_{yz}(y_{12} + \varepsilon, z)] = 0 \quad (53)$$

are used. To obtain the normal stress  $\sigma_y = \sigma_y(y, z)$  we consider the next equation of mechanical equilibrium

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0. \quad (54)$$

Integration of Eq. (54) gives

$$\begin{aligned} \sigma_y = & E_2 \left[ \left( \frac{y^2 + y_2^2}{2} - yy_2 \right) \frac{c_2}{c} \frac{d^3s}{dz^3} - \right. \\ & \left. - \frac{1}{2} \left( \frac{y^3 + 2y_2^3}{3} - y_2^2 y \right) \frac{d^4v}{dz^4} \right], \quad (x, y, z) \in B_2, \\ \sigma_y = & E_2 \left[ \left( \frac{y_{12}^2 + y_2^2}{2} - y_{12} y_2 \right) \frac{c_2}{c} \frac{d^3s}{dz^3} - \right. \\ & \left. - \frac{1}{2} \left( \frac{y_{12}^3 + 2y_2^3}{3} - y_2^2 y_{12} \right) \frac{d^4v}{dz^4} \right] + \\ & + E_1 \left[ \left( \frac{y^2 + y_{12}^2}{2} - yy_{12} \right) \frac{c_1}{c} \frac{d^3s}{dz^3} - \right. \\ & \left. - \frac{1}{2} \left( \frac{y^3 + 2y_{12}^3}{3} - y_{12}^2 y \right) \frac{d^4v}{dz^4} \right] - (y - y_{12}) \left( \frac{\partial \tau_{yz}}{\partial z} \right)_{y=y_{12}}, \quad (56) \\ & (x, y, z) \in B_1. \end{aligned}$$

Here, we use the stress boundary condition

$$\sigma_y(y_2, z) = 0, \quad (57)$$

and stress continuity condition of  $\sigma_y$  at  $y = y_{12}$

$$\lim_{\varepsilon \rightarrow 0} [\sigma_y(y_{12} - \varepsilon, z) - \sigma_y(y_{12} + \varepsilon, z)] = 0. \quad (58)$$

Integration of Eq. (49) leads to next equation

$$\tau_{yz}(y_1, z) - \tau_{yz}(y_2, z) + \frac{\partial}{\partial z} \int_{y_2}^{y_1} \sigma_z dy = 0, \quad (59)$$

that is

$$\tau_{yz}(y_1, z) = -\frac{1}{b} \frac{\partial N}{\partial z} = 0. \quad (60)$$

By the same method from Eq. (54) we obtain

$$\sigma_y(y_1, z) - \sigma_y(y_2, z) + \frac{\partial}{\partial z} \int_{y_2}^{y_1} \tau_{yz} dy = 0, \quad (61)$$

that is

$$\sigma_y(y_1, z) = -\frac{1}{b} \frac{\partial V}{\partial z} = 0. \quad (62)$$

Eqs. (60) and (62) show that the stress boundary conditions for  $\tau_{yz}$  and  $\sigma_y$  at  $y = y_1$  are satisfied. In the following we prove that

$$\tau_{yz}(y_{12}, z) = \frac{Q(z)}{b} = \frac{ks(z)}{b}. \quad (63)$$

Starting from Eq. (50) we can write

$$\begin{aligned} \tau_{yz}(y_{12}, z) = & -E_2 \left[ (y_{12} - y_2) \frac{c_2}{c} \frac{d^2s}{dz^2} - \frac{1}{2} (y_{12}^2 - y_2^2) \frac{d^3v}{dz^3} \right] = \\ = & -E_2 \left[ \frac{c_2 h_2}{c} \frac{d^2s}{dz^2} - c_2 h_2 \frac{d^3v}{dz^3} \right] = \\ = & -\frac{E_2 A_2 c_2}{b c} \left[ \frac{d^2s}{dz^2} - c \frac{d^3v}{dz^3} \right] = -\frac{E_2 A_2 c_2}{b c} \frac{k}{\langle AE \rangle_{-1}} s(z) = \\ = & \frac{E_1 A_1 E_2 A_2}{\langle AE \rangle \langle AE \rangle_{-1}} \frac{Q(z)}{b} = \frac{Q(z)}{b} \end{aligned} \quad (64)$$

according to Eq. (63). Here, Eqs. (2,3,4,5) and Eqs. (26), (50) have been used to prove the validity of Eq. (64).

**NUMERICAL EXAMPLE**

The following data are used in the numerical example (Figure 1):

$$\begin{aligned} b = 0.03 \text{ m}, \quad h_1 = 0.01 \text{ m}, \quad h_2 = 0.03 \text{ m}, \quad E_1 = 1.22 \times 10^{11} \text{ Pa}, \\ E_2 = 8 \times 10^{10} \text{ Pa}, \quad L = 1.5 \text{ m}, \quad \alpha_1 = 2.8 \times 10^{-6} \text{ 1/K}, \\ \alpha_2 = 1.43 \times 10^{-5} \text{ 1/K}, \quad T = 200 \text{ K}, \quad k = 60 \times 10^6 \text{ Pa}. \end{aligned}$$

Figure 4 shows the graph of deflection function and the graph of slip function is illustrated in Figure 5.

The stresses  $\sigma_z = \sigma_z(y, z)$ ,  $\tau_{yz} = \tau_{yz}(y, z)$  and  $\sigma_y = \sigma_y(y, z)$  for some cross section ( $z = L/4$ ,  $z = L/3$ ,  $z = L/2$ ) are shown in Figures 6, 7 and 8.

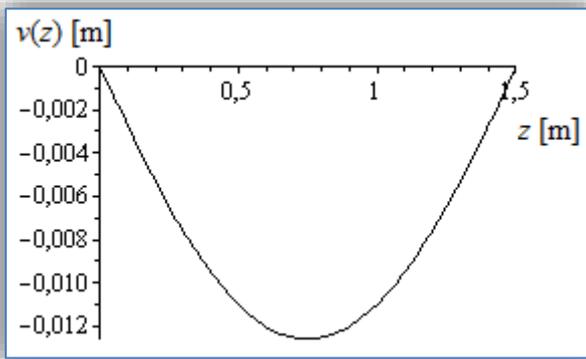


Figure 4. The graph of  $v = v(z)$

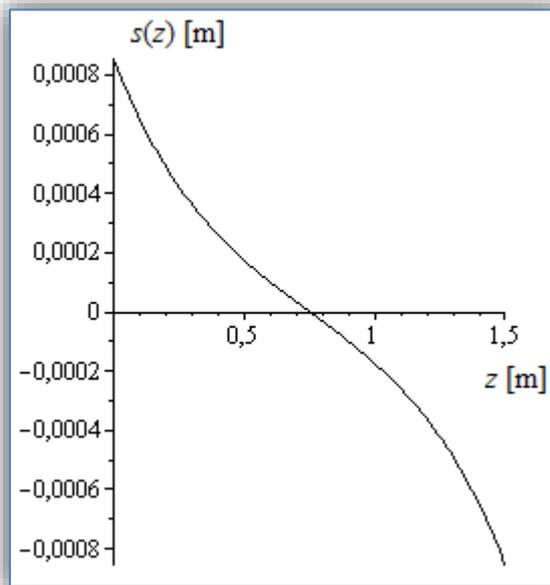


Figure 5. The graph of  $s = s(z)$

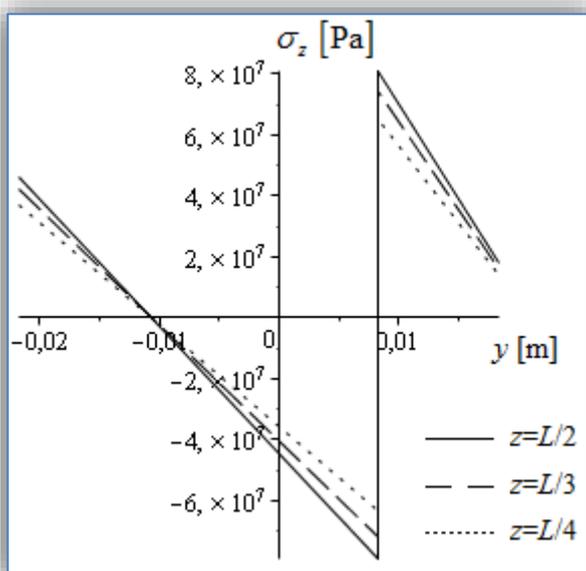


Figure 6. Plots of  $\sigma_z = \sigma_z(y, z)$

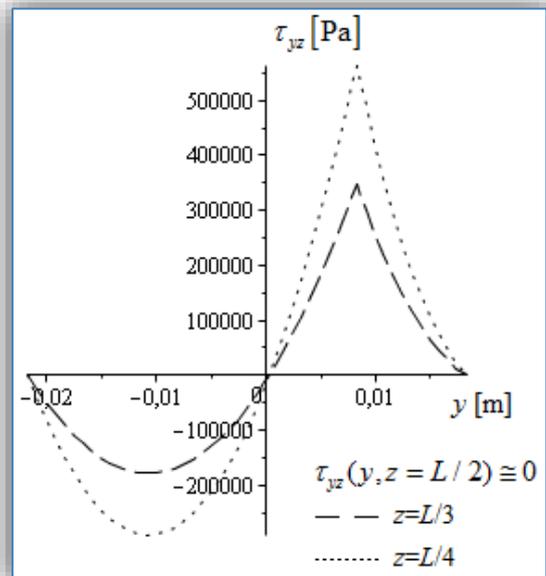


Figure 7. Plots of  $\tau_{yz} = \tau_{yz}(y, z)$

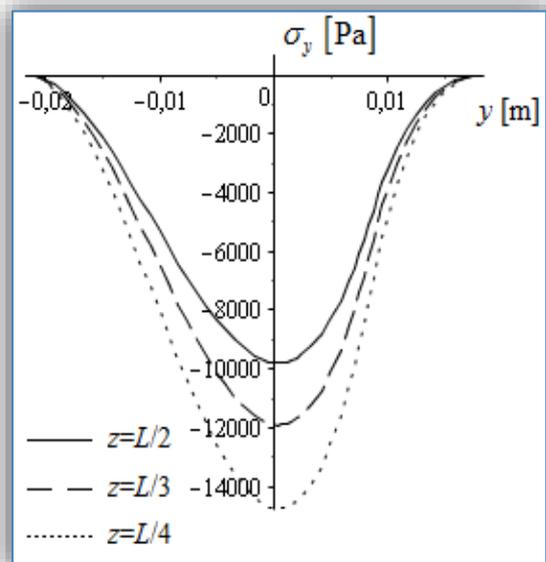


Figure 8. Plots of  $\sigma_y = \sigma_y(y, z)$

### CONCLUSIONS

In this paper the elastic bimetallic beam with flexible shear connection is analyzed. The applied thermal load is caused by a uniform temperature change.

An analytical method, which is based on slip-deflection formulation, is proposed to get the displacements and stresses.

A numerical example illustrates the application of method developed. Numerical solutions derived by this analytical method can be used as benchmark solutions for solutions obtained by other methods.

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