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FLUID FLOW INDUCED BY A MOBILE PROFILE WITH NON-CONSTANT CIRCULATION

Abstract:
This paper deals with an approach of the inviscid 2-dimensional fluid flow induced by the rototranslation of a profile (with a cuspidal point) in the fluid mass, by accepting a non-constant circulation $\Gamma(t)$ around the profile, i.e., multiformity for the pressure field. Some aerodynamic characteristics of a flow induced by an oscillatory motion of a Joukovski profile are calculated.

Keywords:
Non-constant circulation; Joukovski J010 Airfoil

GENERALITIES ON THE UNSTEADY FLOW INDUCED BY A MOBILE PROFILE

Let us consider the two-dimensional unsteady irrotational flow of an inviscid incompressible fluid, induced by the motion of a (wing) profile c with a cuspidal point at the trailing edge, the fluid being supposed at rest at infinity. The contour of the profile c is a simple, closed rectifiable curve ∂c while the exterior mass forces are neglected.

By considering a fixed of rectangular coordinates Ox_1y_1 together with a mobile frame Oxy linked to the mobile profile, we denote at every time t, by

$$\alpha(t) = \widehat{(O_1x_1, Ox)},$$

by $z_0 = x_0 + iy_0$ the affix of the origin of the system Oxy which has the velocity $\vec{v}_0(u_0, v_0)$, by $\vec{r} = x\vec{i} + y\vec{j} \equiv x_1\vec{i}_1 + y_1\vec{j}_1$ the position vector $\vec{r}$ of an arbitrary point $M \in ext(c)$ while $(\vec{i}, \vec{j})$ and $(\vec{i}_1, \vec{j}_1)$ are the unit vectors of the mobile and fixed coordinates system respectively, by $\vec{\omega}(0,0,\omega)$ the instantaneous rotation of the mobile frame. We can write that the absolute velocity, $\vec{v}$, of a fluid particle, located at $M(\vec{r})$, is given by $\vec{v} = u\vec{i} + v\vec{j} = \vec{v}_t + \vec{v}_r$,

where $\vec{v}_t = \vec{v}_0 + \vec{\omega} \times \vec{r}$ is the transport velocity while $\vec{v}_r = u_r\vec{i} + v_r\vec{j} \equiv \dot{x}\vec{i} + \dot{y}\vec{j}$ is the relative velocity of the fluid particle M.

Concerning the fluid flow equations (within the mobile frame Oxy), denoting by p, ρ, V the pressure, the mass density and the magnitude of the absolute velocity respectively, they are

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \operatorname{grad}\left(\dfrac{1}{2}V^2 - \vec{v}\cdot\vec{v}_t\right) + \dfrac{1}{\rho}\operatorname{grad} p = 0 \\ \operatorname{div}\vec{v} = 0 \\ \operatorname{rot}\vec{v} = 0 \end{cases}.$$

In what follows we will accept the multiformity of the pressure field which leads to a period of the pressure around the profile and implicitly to a non-constant circulation $\Gamma(t)$ around the contour ∂c.

By introducing the complex potential $f(z;t) = \varphi(x,y;t) + i\psi(x,y;t)$ and the complex velocity $w = f'(z) = u - iv$, where $u = \dfrac{\partial\varphi}{\partial x} = \dfrac{\partial\psi}{\partial y}$ and $v = \dfrac{\partial\varphi}{\partial y} = -\dfrac{\partial\psi}{\partial x}$, for solving the above proposed *flow problem, we are led to the following boundary value problem for the complex potential:*

Find the function $f : d \equiv \text{ext}(c) \to \mathbf{C}$ *so that*

1. $f(z;t)$ *is holomorphic in the unbounded domain* $\underline{d}$, $\forall t > 0$,

2. $w(z,t) = f'(z,t)$ *is a uniform holomorphic function in* $\underline{d}$ *and* $\lim_{|z|\to\infty} w(z) = 0$,

3. $\text{Im}\{f(z,t)\} \equiv \psi$ *is continuous on* ∂c *and*

$$\psi(x,y;t)\big|_{\partial c} = u_0(t)\cdot y - v_0(t)\cdot x - \omega(t)\cdot\frac{x^2+y^2}{2}.$$

In order to solve this problem we will follow the classical conformal mapping technique. Denoting by

$$H : D \to d,\ z = H(Z) = a_\infty Z + a_0 + \frac{a_1}{Z} + \frac{a_2}{Z^2} + \ldots$$

the conformal mapping which applies the exterior D *of a circumference* C *of centre* O *and radius* R, *i.e.,* $C(O,R)$, *of the plane* OXY $(Z = X + iY)$, *onto the physical flow domain* $\underline{d}$, *the transformed complex potential* $F(Z;t) = f(H(Z)) = \Phi(X,Y;t) + i\Psi(X,Y;t)$ *becomes a holomorphic function in* $D \equiv \text{ext}(C)$ *which is regular at infinity and whose imaginary part* $\Psi = \text{Im}\{F(Z;t)\}$ *on the circumference satisfies*

$$\Psi\big|_C = u_0 y(X,Y) - v_0 x(X,Y) - \omega\frac{x^2(X,Y)+y^2(X,Y)}{2}\Bigg|_{\partial C}$$

where $x(X,Y) + iy(X,Y) = H(X+iY)$.

But this exterior Dirichlet problem for a circle could be solved by considering the Schwartz-Villat formula for determining the flow with circulation $\Gamma(t)$, *i.e.,*

$$F(Z;t) = -\frac{1}{\pi}\int_C \Psi(\varsigma)\frac{d\varsigma}{\varsigma - Z} + \frac{\Gamma(t)}{2\pi i}\text{Log}\frac{Z}{R}.$$

Assuming that the cuspidal point at the trailing edge of our profile with the affix $b = x_b + iy_b$ *corresponds to the point* $Z = R$ *of the plane* Z, *the involved conformal mapping could be represented in the form*

$$z = H(Z) = b + (Z-R)^2\cdot q(Z),\ q(R) \neq 0,$$

and its derivative by $H'(Z) = (Z-R)q_1(Z)$ *where*

$$q_1(Z) = 2q(Z) + (Z-R)q'(Z).$$

Following now the Couchet way [2] for a complete determining of the complex potential of the flow, we will express the unknown complex potential $F(Z)$ *under the form:*

$$F(Z;t) = u_0(t)F_1(Z) + v_0(t)F_2(Z) + \omega(t)F_3(Z) + \Gamma(t)F_4(Z)$$

where $F_1(Z) = H(Z) - a_\infty Z - a_0 - \dfrac{a_\infty R^2}{Z}$,

$$F_2(Z) = -i\left[H(Z) - a_\infty Z - a_0 - \frac{a_\infty R^2}{Z}\right],$$

$$F_3(Z) = \frac{1}{2\pi}\int_C \frac{r^2(\varsigma)}{\varsigma - Z}d\varsigma,$$

where $r^2(\varsigma) = x^2(\varsigma) + y^2(\varsigma) = H(\varsigma)\cdot\overline{H(\varsigma)}$ *and*

$$\Omega(Z) = \frac{1}{2i}F_3{}'(Z) = \frac{1}{4\pi i}\int_C \frac{dr^2(\varsigma)}{\varsigma - Z},$$

$$F_4(Z) = \frac{1}{2\pi i}\text{Log}\frac{Z}{R}.$$

Considering the complex velocity

$$w(z;t) = \frac{df(z;t)}{dz} = \frac{dF(Z;t)}{dZ}\cdot\frac{1}{H'(Z)},$$

by imposing the Joukovski rule (at the image of the cuspidal point) $\left.\dfrac{dF(Z;t)}{dZ}\right|_{Z=R} = 0$, *we get the necessary value of the circulation* $\Gamma(t)$, *namely,* $\Gamma(t) = 4\pi R(a_\infty v_0 + \Omega\cdot\omega)$.

The velocity expression $(u - iv)$ *becomes* $u_0K + v_0L + \omega M$, *where*

$$K = 1 - \frac{a_\infty(Z^2 - R^2)}{Z^2\cdot H'(Z)},$$

$$L = -i\left[1 - \frac{a_\infty}{Z^2\cdot q_1(Z)}(Z-R)\right],$$

$$M = \frac{2i}{Zq_1(Z)}\,\frac{Z\Omega(Z) - R\Omega(R)}{Z-R}.$$

The velocity value at the trailing edge $Z = R$ *is*

$$w(b) = u_0\left[1 - \frac{2a_\infty}{Rq_1(R)}\right] - iv_0 + \omega\frac{2i(\Omega + R\Omega')}{Rq_1(R)},$$

where $\Omega = \Omega(R)$ *and* $\Omega' = \Omega'(R)$.

THE CASE OF JOUKOVSKI TYPE PROFILE

Let us now consider the particular case of a Joukovski type profile. To make precise by using a conformal mapping of the type

$$z = H(Z) = a_0 + Z + \frac{(1-X_0)^2}{Z - X_0},\ 0 < X_0 < 1,$$

the image of the circumference $C(O,R)$ *of the plane* Z *becomes the considered Joukovski profile whose rototranslation induces the fluid flow. We denote by AB* $(A(a,0)$, $B(b,0)$, $a = H(-1)$, $b = H(1)$*), the profile chord, its trailing edge (cuspidal point) being* $z = b = H(1)$.

Using the development

$$H(Z) = Z + a_0 + \frac{(1-X_0)^2}{Z} + \frac{X_0(1-X_0)^2}{Z^2} + \frac{X_0{}^2(1-X_0)^2}{Z^3} + \ldots$$

we find $a_\infty = 1$, $a_1 = (1-X_0)^2$ *and*

$$H'(Z)=\frac{Z+1-2X_0}{(Z-X_0)^2}(Z-1).$$

We also obtain the expressions

$$\Omega(Z)=\frac{1}{2}\left[\frac{a_0}{Z^2}+\frac{a_0(1-X_0)^2}{(Z-X_0)^2}+\frac{(1-X_0)^2(2Z-X_0)}{Z^2(Z-X_0)^2}+\frac{X_0(1-X_0)^3}{(1+X_0)(Z-X_0)^2}\right]$$

$$\Omega=\Omega(1)=a_0+\frac{1+X_0-X_0^2}{1+X_0}.$$

The circulation and the complex potential are, respectively,

$$\Gamma=4\pi\left[v_0+\omega\left(\frac{a_0}{2}+\frac{1+X_0-X_0^2}{1+X_0}\right)\right]$$

$$F(t,Z)=u_0\left[\frac{(1-X_0)^2}{Z-X_0}-\frac{1}{Z}\right]-iv_0\left[\frac{(1-X_0)^2}{Z-X_0}+\frac{1}{Z}\right]-$$
$$-i\omega\left[\frac{a_0}{Z}+\frac{a_0(1-X_0)}{Z-X_0}+\frac{(1-X_0)^2}{Z(Z-X_0)}+\frac{X_0(1-X_0)^3}{(1+X_0)(Z-X_0)}\right]-$$
$$-2i\left[v_0+\omega\left(\frac{a_0}{2}+\frac{1+X_0-X_0^2}{1+X_0}\right)\right]\cdot \mathrm{Log}Z.$$

Consequently the complex velocity $u-iv=u_0K+v_0L+\omega M$, *where*

$$K(Z)=1-\frac{Z+1}{Z}\cdot\frac{(Z-X_0)^2}{Z+1-2X_0},$$

$$L(Z)=-i\left[1-\frac{(Z-X_0)^2(Z-1)}{Z^2(Z+1-2X_0)}\right],$$

$$M(Z)=\frac{2i}{Z}\cdot\frac{(Z-X_0)^2}{Z+1-2X_0}\cdot\frac{Z\Omega(Z)-\Omega(1)}{Z-1}$$

$$\frac{Z\Omega(Z)-\Omega(1)}{Z-1}=-\frac{a_0}{2}\left[\frac{1}{Z}+\frac{Z-X_0^2}{(Z-X_0)^2}\right]+$$

$$\frac{2(X_0^2-X_0-1)Z^2-(X_0^4+X_0^3-3X_0^2-3X_0+2)Z+X_0(1-X_0)^2(1+X_0)}{2(1+X_0)Z(Z-X_0)^2}.$$

Putting then $Z=1$ *we have* $K=X_0$, $L=-i$,

$$M=i(1-X_0)\cdot\lim_{Z\to 1}\frac{Z\Omega(Z)-\Omega(1)}{Z-1}=-i(2+a_0-X_0)$$

and the value of the velocity at the trailing edge is $u-iv|_{Z=1}=X_0u_0-iv_0-i(2+a_0-X_0)\omega$.

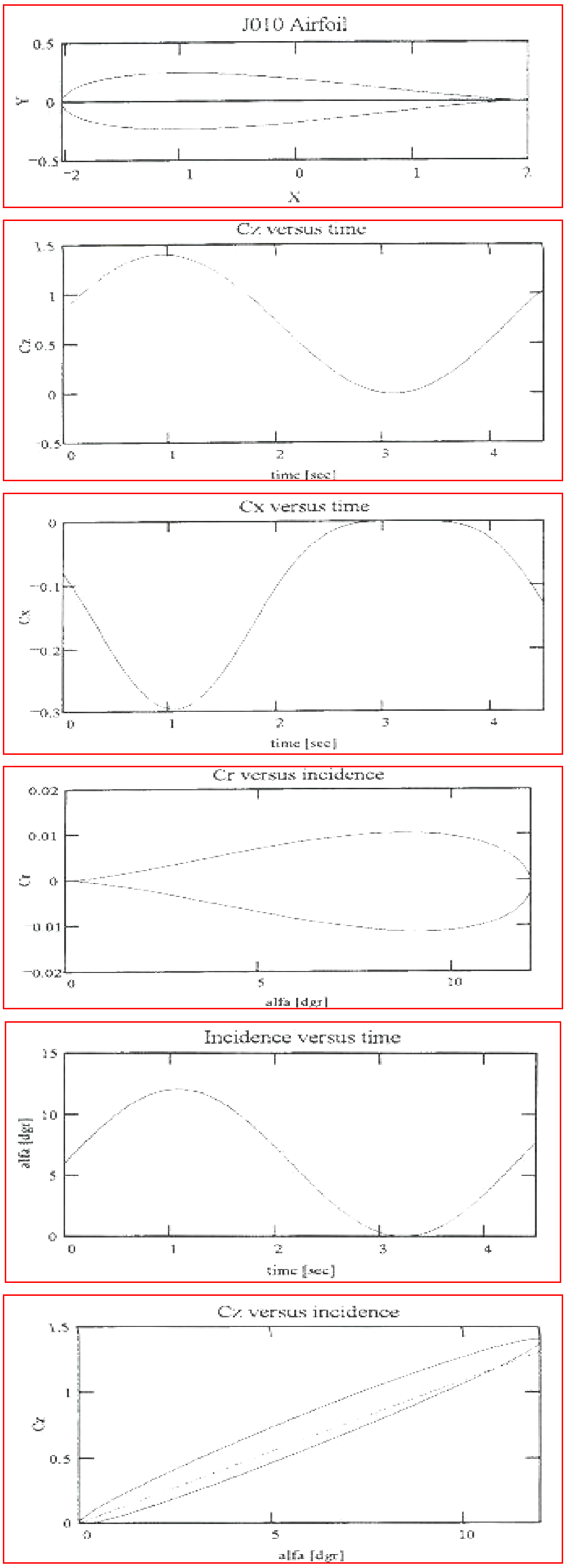
The non-constant circulation $\Gamma(t)$ *implies a multiform pressure field, the pressure admitting a period around the profile. So, from Bernoulli formula we obtain*

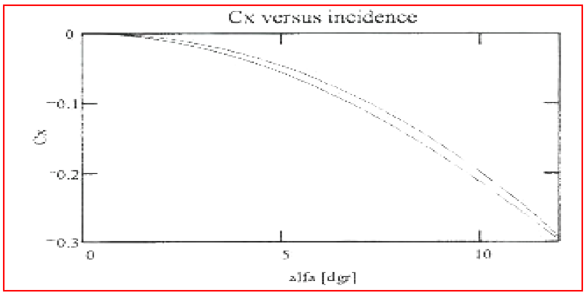
$$\int_C dp=-\rho\frac{d\Gamma}{dt}=-4\pi\rho\left(\frac{dv_0}{dt}+\Omega\frac{d\omega}{dt}\right).$$

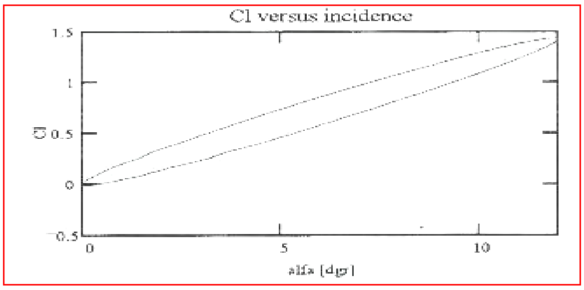
Mach = 0.112, k = 0.077

a = 6 + 10 * sin (1.461 * t)

FIGURE 1. Aerodynamic coefficients for oscillating Joukovski J010 airfoil







According to the Blasius-Chapligin formulas and imposing the Joukovski rule we obtain the pressure resultant components and the moment magnitude, i.e.

$$R_x = 2\pi\rho\begin{bmatrix} -2v_0^2 + (B_1 - 2a_0 - 2\Omega)v_0\omega \\ +(C_1 - 2a_0\Omega)\omega^2 + A_1\dfrac{du_0}{dt}\end{bmatrix}$$

$$R_y = 2\pi\rho\begin{bmatrix} 2u_0v_0 + (A_1 + 2\Omega)u_0\omega - (4 + B_1 - 2X_0)\dfrac{dv_0}{dt} \\ -(C_1 + 4\Omega - 2X_0\Omega)\dfrac{d\omega}{dt}\end{bmatrix}$$

$$\mathcal{M}_0 = -\rho\begin{bmatrix} 2\pi(A_1 + B_1 - 2a_0)u_0v_0 - 4\pi a_0\Omega u_0\omega \\ +(2\pi C_1 + Q)\dfrac{dv_0}{dt} + (\pi P + Q\Omega)\dfrac{d\omega}{dt}\end{bmatrix},$$

where $A_1 = \dfrac{A}{2i\pi}$, $B_1 = \dfrac{B}{2\pi}$, $C_1 = \dfrac{C}{2\pi}$ *and*

$$A = 2i\pi\left[(1 - X_0)^2 - 1\right] + i\frac{4\pi X_0}{(1 + X_0)^2} = -2i\pi\frac{X_0^2(3 - X_0^2)}{(1 + X_0)^2}$$

$$B = 2\pi\left[(1 - X_0)^2 - 1\right] - \frac{4\pi X_0}{(1 + X_0)^2} = 2\pi\frac{X_0^4 - X_0^2 + 2}{(1 + X_0)^2}$$

$$C = 2\pi a_0\left[(1 - X_0)^2 + 1\right] + 2\pi X_0\frac{(1 - X_0)^3}{1 + X_0}$$

$$-\frac{4\pi X_0}{(1 + X_0)^2}\left[a_0 - \frac{(1 - X_0)^2}{2(1 + X_0)}\right]$$

$$Q = 4\pi a_0 + 4\pi a_0(1 - X_0) + 4\pi(1 - X_0) + 4\pi X_0\frac{(1 - X_0)^2}{1 + X_0}$$

The final diagrams of this paper (figure 1) present the numerical results calculated with the above method for the symmetrical Joukovski J010 airfoil in an oscillating motion defined by the oscillation angle $\alpha(t) = \alpha_0 + A_m \cdot \sin\dfrac{kV}{a}t$,

where $a = |AB|/2$, $k = 0{,}077$, $V = 340 \cdot \mathrm{Mach}$.

We denoted $C_x = \dfrac{R_x}{\rho a V^2}$, $C_z = \dfrac{R_y}{\rho a V^2}$, $C_r = \dfrac{R_{x1}}{\rho a V^2}$, $C_l = \dfrac{R_{y1}}{\rho a V^2}$, *where* $R_{x1} = R_x\cos\alpha + R_y\sin\alpha$ *and* $R_{y1} = -R_x\sin\alpha + R_y\cos\alpha$.

The continuous line represents our results and the dotted line in the diagram "C_z versus incidence", represents the results obtained in the paper [4].

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